

Bubbles of nothing and supersymmetric compactifications

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Abstract. We investigate the non-perturbative stability of supersymmetric compactifications with respect to decay via a bubble of nothing. We show examples where this kind of instability is not prohibited by the spin structure, i.e., periodicity of fermions about the extra dimension. However, such “topologically unobstructed” cases do exhibit an extra-dimensional analog of the well-known Coleman-De Luccia suppression mechanism, which prohibits the decay of supersymmetric vacua. We demonstrate this explicitly in a four dimensional Abelian-Higgs toy model coupled to supergravity. The compactification of this model to $M_3 \times S_1$ presents the possibility of vacua with different windings for the scalar field. Away from the supersymmetric limit, these states decay by the formation of a bubble of nothing, dressed with an Abelian-Higgs vortex. We show how, as one approaches the supersymmetric limit, the circumference of the topologically unobstructed bubble becomes infinite, thereby preventing the realization of this decay. This demonstrates the dynamical origin of the decay suppression, as opposed to the more familiar argument based on the spin structure. We conjecture that this is a generic mechanism that enforces stability of any topologically unobstructed supersymmetric compactification.

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1 Introduction

Higher dimensional theories have been extensively studied in the last few decades as possible extensions of the Standard Model. Incorporating these extra dimensions into the fundamental theory adds new degrees of freedom to the low energy dynamics. One must first demonstrate the existence of a perturbatively stable vacuum in such a theory before one can infer any new physics. This is normally achieved by a compactification mechanism that generates a potential for these new degrees of freedom. Our vacuum might be a minimum in such a potential, and therefore perturbatively stable. Finding stable vacua compatible with current

observations is one of the major challenges facing all fundamental higher dimensional theories, such as String Theory.

It is far from obvious that there should exist a unique vacuum in the effective 4D description of these higher dimensional theories, and often one encounters multiple minima within the same compactification potential. This opens up the possibility of a new type of instability due to the presence of large quantum mechanical fluctuations about the original vacuum. Such tunneling processes were first discussed in a series of papers [1, 2] in the context of a 4D field theory. The results of these papers are that a non-perturbative instability of false vacua occurs through the nucleation of a bubble, whose interior consists of the lower energy vacuum. The bubble wall is a field configuration interpolating between the higher energy initial vacuum (parent vacuum) and the final one (daughter vacuum). In many cases, one can reduce this problem to the simpler version within the so-called thin wall approximation, where one can assume that the bubble is spherical and made of a thin solitonic domain wall (of fixed tension) characterized by just its radius. Once the bubble is formed by the quantum mechanical tunneling process, the pressure difference across the wall accelerates the bubble, making it grow and gain kinetic energy by converting arbitrarily large regions of spacetime into the new lower energy density vacuum. Clearly it is important to estimate the rate of this instability if our universe is to be described by any of the theories susceptible to vacuum decay. One can calculate the rate of this decay by the use of instanton methods and the computation of the action of the appropriate Euclidean classical solutions. This was done in [2] and further developed in the thin wall approximation for the case of a scalar field potential coupled to gravity in [3]. After coupling the theory to gravity, one discovers that there are now cases where the tunneling probability is completely suppressed. Certain Minkowski and Anti-deSitter false vacua are exactly stable even at the non-perturbative level. The reason for this is that all saddle points of their Euclidean action correspond to bubbles of infinite circumference, which thus have infinite action. The nucleation rate is exponentially suppressed by the Euclidean action, and therefore vanishes. This enforces the stability of the parent vacuum, making such states interesting starting points for searches for realistic vacua in theories beyond the Standard Model.

Another common ingredient in many extensions of the Standard Model is supersymmetry. In particular, many higher dimensional theories also incorporate supersymmetry (such as in string theory). It is therefore natural to consider compactifications that preserve some supersymmetry. This is not only interesting from the point of view of phenomenology, but also makes the question of stability a much simpler one to study, at least perturbatively [4].

At the non-perturbative level the question of stability becomes much more interesting. This was studied in the context of a simple $d = 4$, $\mathcal{N} = 1$ supergravity model in [5]. The results of that paper indicate that indeed the supersymmetric vacua are stabilized by the previously mentioned Coleman-De Luccia suppression mechanism. The conditions enabling suppression are enforced in that case by the form of the potential: in a theory with distinct supersymmetric vacua, there is a solution containing a domain wall interpolating between them. This domain wall is static and has infinite area. One way to understand Coleman-De Luccia suppression is to think about the required tension of the bubble wall that one would need in a thin wall description of the decay. For sufficiently low tension, the Euclidean action of the thin wall bubble always has a saddle point for a certain critical value of bubble radius, describing the size of the bubble at nucleation. As the tension is increased, the critical radius grows, and at a certain finite value of tension, the critical radius diverges. The static domain wall described above is described precisely by this saddle point, and the infinite area is due

to the domain wall having the tension corresponding to a divergent critical radius.

From this analysis one concludes that there is a limiting tension above which the decay process cannot happen. Interestingly, supersymmetry imposes a lower bound on the tension of the wall interpolating between supersymmetric vacua, so it is clear that the bubble decay process cannot occur if this lower bound on tension is at or above the tension corresponding to an infinite critical radius. Furthermore, one can show that the limiting case where these two values of tension coincide preserves part of the original supersymmetry.

These arguments seem to suggest that models of compactification which can be described by a supersymmetric theory would be stable if their compactification preserves part of the supersymmetry. However, there may be new instabilities of a higher dimensional theory that are not described in the context of a 4D low energy effective theory and therefore one would have to take all the previous considerations with a little bit of caution. An example of such an instability was demonstrated some time ago by Witten [6]. In the simplest example of a 5D Kaluza-Klein compactification to $M_4 \times S^1$, he was able to explicitly construct an instanton for the decay by the formation of a *bubble of nothing*. This instanton describes the formation of a bubble where the extra dimensions pinch off, disappearing, signaling the end of spacetime in this region, hence the name bubble of nothing. Viewed from a 4D perspective, these solutions would be singular¹, so it is hard to argue their existence or validity on the basis of a pure 4D theory. Nevertheless, these potential new instabilities exist and one would wonder if this could lead to the decay of some supersymmetric compactifications. The original paper by Witten [6] already contains clues regarding this subject, which lead to the conclusion that it would be impossible for a supersymmetric compactification with a circle extra dimension to decay in this way. The argument is quite simple. The 5D Kaluza-Klein theory allows for a supersymmetric extension including fermionic degrees of freedom, but its compactification would only preserve some supersymmetry if these fermionic modes are periodic around the extra dimension. On the other hand, the instanton solution that allows the decay and disappearance of the extra-dimension (into nothing) forces the situation with anti-periodic fermions, so it is clear that one would be in a different sector of the theory if one starts with a supersymmetric compactification and this decay would not be possible².

In this paper we want to investigate how generic this argument is. In particular, the justification for stability of supersymmetric compactifications in Witten's argument is entirely based on the spin structure of the theory and the instanton solution and it seems hard to generalize it to other internal spaces. Furthermore, the reasoning for the tunneling suppression is also quite different in nature from the one described earlier in the $\mathcal{N} = 1$ supergravity scalar theory that relies on a dynamical mechanism first identified in field theory by Coleman and De Luccia [3].

The main idea of this paper is to look for the simplest example of a supersymmetric theory where the argument for stability based on the spin structure cannot be used and investigate in this case the possible existence of a bubble of nothing instability. In the following, we will show that there are indeed some supersymmetric compactifications that allow for the same spin structure as the bubble of nothing geometry, therefore circumventing Witten's argument on stability. However, we will show that in these cases, the stability of the compactification is preserved by the Coleman-De Luccia suppression mechanism, where the nucleated bubble would need to be infinitely large and the decay would therefore be completely suppressed. Hence our conclusions are that, in fact, supersymmetric compactifications remain

¹See for example the discussion in [7].

²For a discussion of the pre-factor of the decay probability in this context see [8].

stable but that the reason for this in some cases may be different from what was originally envisioned in [6].

The plan of the paper is the following. We describe in section 2 the simple model of $d = 4$, $\mathcal{N} = 1$ supergravity that we will consider. We show in section 3 how one can compactify this theory on a circle down to three dimensions, analogous to the usual Kaluza-Klein model. We also show in this section the conditions required to obtain a supersymmetric compactification of this model and study the spin structure of those vacuum solutions. We discuss in section 4 the kind of instanton solutions one would need in order to describe the decay of these compactified vacuum states and their relation to the original bubbles of nothing in the thin wall approximation. In section 5, we present our numerical approach to study these instanton solutions in the supergravity model presented earlier. In section 6 we compare analytic and numerical solutions within the thin wall regime and carefully investigate their limit as the initial state becomes supersymmetric. In section 7 we present generic numerical solutions. Finally we conclude with some remarks in section 8.

2 The model

As we mentioned in the introduction, we would like to study a model whose compactification on a circle allows for anti-periodic fermionic boundary conditions, while still preserving part of its supersymmetry. We will show in this paper that we can find a compactification with these characteristics within the Abelian-Higgs model coupled to $\mathcal{N} = 1$ supergravity in $3 + 1$ dimensions.³ This model has been considered in the literature mainly in the context of cosmic string solutions [9, 10] and we will see later on that these solutions also play a role in our current discussion.⁴

The model describes the dynamics of a complex scalar field ϕ with Kähler potential $K(\phi, \bar{\phi}) = \bar{\phi}\phi$ minimally coupled to a U(1) gauge field A_μ . The superpotential is taken to be $W(\phi) = 0$, and the gauge kinetic function is $f(\phi) = 1$. With these choices, the bosonic part of the action is the well known Einstein-Abelian-Higgs model

$$S_{\text{bos}} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - D_\mu \bar{\phi} D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{e^2}{2} (\eta^2 - \phi \bar{\phi})^2 \right], \quad (2.1)$$

where the gauge covariant derivative is defined by $D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the U(1) field strength. We also introduce for future reference the dimensionless parameter $\gamma \equiv \kappa^2 \eta^2$, which controls, as we will see later on, the gravitational effects of the typical energy scale of this theory.

The full supergravity model also involves the gravitino field, ψ_μ , and the fermionic partners of the chiral and gauge fields, χ and λ , respectively.⁵ It is invariant under the *local*

³We make this choice in order to simplify the model and to use the well known 4D supersymmetric notation, but we do expect the results of this paper to apply to more general models. It would be interesting to look for generalizations of this idea to a truly higher dimensional model or models with other matter content.

⁴In this paper, we will follow the conventions in [11]. In particular we use the Minkowski metric with signature $(-, +, +, +)$, and we work with the units $c = \hbar = 1$, so that the reduced Planck mass reads $\kappa^2 \equiv M_P^{-2} = 8\pi G$.

⁵The gravitino is usually written as a Majorana spinor ψ_μ , but sometimes it is convenient to split it into its complex chiral parts, $\psi_{\mu L} = \frac{1}{2}(1 + \gamma_5)\psi_\mu$, and $\psi_{\mu R} = \frac{1}{2}(1 - \gamma_5)\psi_\mu$. The same notation applies to the gauginos λ and the chiralinos χ .

U(1) gauge transformations

$$\begin{aligned}
\delta_g \phi &= i e \phi \alpha, \\
\delta_g \chi_L &= i e \left(1 + \frac{\eta^2 \kappa^2}{2}\right) \chi_L \alpha, \\
\delta_g \psi_{\mu L} &= -i e \frac{\eta^2 \kappa^2}{2} \psi_{\mu L} \alpha, \\
\delta_g \lambda_L &= -i e \frac{\eta^2 \kappa^2}{2} \lambda_L \alpha,
\end{aligned} \tag{2.2}$$

where α is the gauge parameter. Note that the combination $\xi \equiv e\eta^2$ appearing in the scalar potential also contributes to the charge of all fermions under the local U(1) symmetry. Such a combination can be identified as the Fayet-Iliopoulos (FI) term of $\mathcal{N} = 1$ supergravity, and it is associated to the gauging of the R -symmetry which rotates the supercharges. In order to simplify the notation, we will take the FI term to be a free parameter in the main part of the paper and comment on its quantization in Appendix A. The conclusions of the paper are not affected by this quantization.

In later sections we will discuss the spontaneous breaking of supersymmetry by bosonic backgrounds, so it will be useful to have the form of the supersymmetry transformations for this model. In purely bosonic backgrounds only the supersymmetry transformations of the fermions can be non-vanishing, which read

$$\delta \psi_{\mu L} = \mathcal{D}_\mu \epsilon_L = (\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{i}{2} A_\mu^B) \epsilon_L, \tag{2.3}$$

$$\delta \chi_L = \frac{1}{\sqrt{2}} \gamma^\mu D_\mu \phi \epsilon_R, \tag{2.4}$$

$$\delta \lambda = \frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu} \epsilon + \frac{i}{2} e (\eta^2 - \phi \bar{\phi}) \gamma_5 \epsilon, \tag{2.5}$$

up to terms cubic in the fermions.⁶ Here ϵ is the parameter of the local supersymmetry transformations, and the composite U(1) connection A_μ^B is given by

$$A_\mu^B = \frac{1}{2} i \kappa^2 [\phi D_\mu \bar{\phi} - \bar{\phi} D_\mu \phi] + e \eta^2 \kappa^2 A_\mu. \tag{2.6}$$

3 Supersymmetric Kaluza-Klein compactification on S^1

3.1 Generalized Kaluza-Klein compactification

The model described by the action (2.1) admits a Kaluza-Klein type of vacuum, where one of the spatial dimensions is compactified on a circle, $\mathbb{M}_4 \rightarrow \mathbb{M}_3 \times S^1$, and the matter fields are in their vacuum, namely

$$\phi \bar{\phi} = \eta^2, \quad D_\mu \phi = 0, \quad \psi_\mu = \chi = \lambda = 0, \tag{3.1}$$

$$ds^2 = -dt^2 + dz^2 + dr^2 + R^2 d\theta^2. \tag{3.2}$$

Here the coordinates $t, z, r \in \mathbb{R}$ parametrize the 3-dimensional Minkowski part of the spacetime, and $\theta \in [0, 2\pi)$ is the angular variable associated to the compact dimension whose physical circumference is $2\pi R$. In order to define the (generalized) Kaluza-Klein theory, it is necessary to specify the boundary conditions⁷ of the fields around the compactified dimension

⁶See [11] for further details.

⁷In general, the fields can be identified with sections on a non-trivial fibre bundle, and the choice of boundary conditions specifies the topology of the bundle [12–17].

[18–20]. The fields need to be periodic only up to a global symmetry of the action, so we can formally write

$$\Phi(\theta + 2\pi) = e^{i\hat{Q}\alpha} \Phi(\theta), \quad (3.3)$$

where we have denoted all fields collectively by Φ , and \hat{Q} is the generator of a global symmetry. Any Lorentz invariant action is always invariant under the \mathbb{Z}_2 symmetry which flips the sign of all fermions. In addition, the supergravity model (2.1) has a $U(1)_c \times U(1)_R$ global symmetry, where the first factor corresponds to the $U(1)$ symmetry associated with the chiral supermultiplet, and the second one is the R -symmetry. Thus, on the spacetime described by the line element (3.2) we can impose boundary conditions of the form

$$\begin{aligned} \phi(\theta + 2\pi) &= e^{i\alpha_c} \phi(\theta), \\ \chi_L(\theta + 2\pi) &= \pm e^{i\alpha_c} e^{i\alpha_R} \chi_L(\theta), \\ \psi_{\mu L}(\theta + 2\pi) &= \pm e^{-i\alpha_R} \psi_{\mu L}(\theta), \\ \lambda_L(\theta + 2\pi) &= \pm e^{-i\alpha_R} \lambda_L(\theta), \end{aligned} \quad (3.4)$$

where $\alpha_c \in [0, 2\pi)$ and $\alpha_R \in [0, 2\pi)$ are the parameters of the global $U(1)_c$ and $U(1)_R$ respectively.

As discussed in [6], in a spacetime with the topology of a bubble of nothing, the spinor fields are uniquely defined and only admit anti-periodic boundary conditions along the compactified dimension:⁸

$$\chi_L(\theta + 2\pi) = -\chi_L(\theta), \quad \psi_{\mu L}(\theta + 2\pi) = -\psi_{\mu L}(\theta), \quad \lambda_L(\theta + 2\pi) = -\lambda_L(\theta). \quad (3.5)$$

This implies that for the arbitrary boundary conditions of eqs. (3.4), the Kaluza-Klein vacuum and the bubble of nothing belong to topologically distinct sectors. This guarantees the generic stability of Kaluza-Klein vacua with respect to this decay channel [6]. In summary, only Kaluza-Klein vacua whose fermions satisfy anti-periodic boundary conditions may decay via the formation of bubbles of nothing.

3.2 Pure vacuum solutions and periodic fermions

For the Kaluza-Klein background to preserve the full supersymmetry of the model, the supersymmetry transformations (2.3), (2.4) and (2.5) must vanish for all values of the parameter ϵ . In a background of the form (3.1), where the matter fields are on a pure vacuum configuration

$$\phi = \eta, \quad A_\mu = 0, \quad \psi_\mu = \chi = \lambda = 0, \quad ds^2 = -dt^2 + dz^2 + dr^2 + R^2 d\theta^2, \quad (3.6)$$

only the gravitino transformation is non-trivial, which reduces to

$$\mathcal{D}_\mu \epsilon_L = \partial_\mu \epsilon_L = 0. \quad (3.7)$$

Then, for this background to preserve supersymmetry the theory must admit a covariantly constant spinor. Such a solution must be globally well defined, meaning that it should be consistent with the boundary conditions (3.4) that we have imposed for the Kaluza-Klein reduction. The solutions to the previous equation are just constant spinor parameters, $\epsilon_L(x^\mu) = \epsilon_L^0$, which are periodic, and thus the background (3.6) can only be supersymmetric

⁸The construction of spinor structures on simply connected spacetimes and on the cylinder (3.6) is discussed in [14, 16, 21–23].

when the boundary conditions for the fermions in the Kaluza-Klein reduction are chosen to be periodic.

In the case of the bubble of nothing, the background spacetime is simply connected and asymptotically approaches a cylinder, where as we described earlier, one must impose that the fermions be antiperiodic as one goes around the extra-dimensional circle. It therefore follows that supersymmetry must be broken in the asymptotic pure KK background state of the bubble of nothing. This is consistent with the results in [24–27] for purely gravitational theories, where it was shown that covariantly constant spinors do not exist in asymptotically conical spacetimes. Indeed, the spacetime of the bubble of nothing is asymptotically conical with deficit angle of 2π , that is, a cylinder.

This relation between supersymmetry and the boundary conditions of the fermions can also be understood intuitively by looking at the mass spectrum of the KK theory [18, 19]. When we perform a Kaluza-Klein reduction in the background (3.6) but with boundary conditions other than periodic, some of the fermions that would be present in the reduced theory acquire masses of the order of the KK scale. As a result, supersymmetry is broken in the dimensionally reduced theory.⁹

This is just another way of stating the result in [6] that the simple supersymmetric vacuum would not be allowed to decay by the formation of a bubble of nothing due to the incompatibility of the spin structures between the supersymmetric compactification state and the bubble of nothing geometry.

3.3 Winding compactifications and antiperiodic fermions

From our discussion in the previous paragraph, we see that imposing that the background (3.6) be supersymmetric introduces a topological obstruction to the formation of bubbles of nothing. However, this topological obstruction is not always present in all possible supersymmetric backgrounds. Indeed, it is easy to see that the simple model (2.1) admits supersymmetric compactifications demonstrating this. Consider the following bosonic background, which is also of the form of eqs. (3.1)-(3.2),

$$\phi = \eta e^{i\theta}, \quad A_\mu = e^{-1} \delta_{\mu\theta}, \quad \psi_\mu = \chi = \lambda = 0, \quad ds^2 = -dt^2 + dz^2 + dr^2 + R^2 d\theta^2. \quad (3.8)$$

The main difference with respect to the vacuum considered before (3.6) is that the gauge vector has non-vanishing vacuum expectation value, so that the configuration has a U(1) Wilson line on the compact direction. Furthermore, in order to satisfy $D_\mu \phi = 0$, the scalar field also needs to wind around the compact direction.

As in the previous example, the only non-trivial supersymmetry transformation is the one of the gravitino,

$$\mathcal{D}_\mu \epsilon_L = (\partial_\theta + \frac{i}{2} \kappa^2 \eta^2) \epsilon_L = 0, \quad \implies \quad \epsilon_L(x^\mu) = e^{-i \frac{\kappa^2 \eta^2}{2} \theta} \epsilon_L^0. \quad (3.9)$$

We can see that, provided the parameters of the theory satisfy

$$\gamma = \kappa^2 \eta^2 = 1, \quad (3.10)$$

it is possible to find a covariantly constant spinor which is consistent with imposing antiperiodic boundary conditions for the fermions. This is just a special case of the result found in [28], showing that covariantly constant spinors may exist in conical spacetimes when they are

⁹See a more detailed description of this point in the Appendix A.

coupled to a $U(1)$ gauge field. Note that the background (3.8) can be consistently interpreted as the asymptotic region of a conical spacetime with deficit angle of 2π , where fermions are necessarily anti-periodic, and therefore the same mechanism ensures that supersymmetry is fully preserved.

Intuitively, the relation between the Wilson line, the winding scalar field and supersymmetry breaking can also be understood as before by looking at how it affects the KK mass spectrum. As was argued in [29], when the configuration has a Wilson line on the S^1 , the whole KK mass spectrum of the fermions coupled to it gets shifted by an amount proportional to the magnitude of the Wilson line. Then, choosing conveniently the expectation value of the vector boson and the couplings, it is possible to tune to zero the masses that would be induced by the non-periodic boundary conditions. As a consequence, the field content left after the reduction is sufficient to form full supermultiplets, as in the case with trivial boundary conditions, and therefore it is possible to obtain a low energy theory invariant under supersymmetry. We describe this argument in more detail in the Appendix A.

Summarizing, the field configuration (3.8) represents a supersymmetric KK compactification which is consistent with anti-periodic fermions on the circle and, in consequence, it has no topological protection against decay via nucleation of bubbles of nothing.

4 The Bubble of nothing geometry

In previous sections we have discussed several compactification scenarios of our model. We now describe the bubble of nothing geometries that would represent the decay of these compactifications to nothing.

4.1 Bubble of nothing for pure vacuum solutions

We start our discussion with a lower dimensional version of the usual bubble of nothing vacuum solution [6], which in 4D is given by the double Wick rotation of the Schwarzschild solution:

$$ds^2 = \rho^2 (-dt^2 + \cosh^2 t d\chi^2) + \left(1 - \frac{\rho_0}{\rho}\right)^{-1} d\rho^2 + \left(1 - \frac{\rho_0}{\rho}\right) d\Theta^2. \quad (4.1)$$

Here χ is an angular coordinate in $[-\pi, \pi)$, $\rho \in [\rho_0, \infty)$ is a radial coordinate, and Θ is a periodic variable which runs from 0 to $2\pi R$, R being the asymptotic radius of the compact KK dimension. The parameter ρ_0 determines the size of the bubble at the time of its formation, $t = 0$. This is a vacuum solution of Einstein's equations.

In order to discuss the geometry of this spacetime it is convenient to introduce a new coordinate system $\{\tau, r, z, \theta\}$, given by

$$t = H_0 \tau, \quad \chi = H_0 z, \quad \Theta = R \theta, \quad (4.2)$$

where $H_0 = \rho_0^{-1}$, which must be assumed positive for now. Note that the angular variable z now takes values in $[-\frac{\pi}{H_0}, \frac{\pi}{H_0})$, and the coordinate θ parametrizing the compact direction runs in $[0, 2\pi)$. The new radial coordinate $r \in [0, \infty)$ is defined implicitly in terms of the differential equation

$$\frac{d\rho}{dr} = \sqrt{1 - \frac{\rho_0}{\rho}}, \quad (4.3)$$

and the boundary condition $\rho(0) = \rho_0$ (or $r(\rho_0) = 0$). That is, the position of the bubble is now given by $r = 0$. In this gauge, the metric takes the form

$$ds^2 = B(r)^2 (-d\tau^2 + \cosh^2(H_0 \tau) dz^2) + dr^2 + C(r)^2 d\theta^2, \quad (4.4)$$

where the metric profile functions $B(r) = H_0 \rho(r)$ and $C(r)$ are determined implicitly by the expressions

$$\begin{aligned} r(B) &= H_0^{-1} \sqrt{(B-1)B} + H_0^{-1} \log \left(\sqrt{B} + \sqrt{B-1} \right), \\ C(r) &= R \sqrt{1 - 1/B(r)}. \end{aligned} \quad (4.5)$$

It is worth pointing out that the bubble of nothing geometry has a characteristic property which is an immediate consequence of the definition of $B(r)$ and the equations (4.3) and (4.5):

$$B'(r) = H_0 \sqrt{1 - B^{-1}} = H_0 C(r)/R. \quad (4.6)$$

Later on we will use this property to determine when a solution is an approximate solution of the vacuum Einstein's equations of the bubble of nothing type. More specifically, when the metric functions of our solutions fulfill (approximately) eq. (4.6), it will mean that the metric configuration resembles that of the pure vacuum bubble of nothing solution.

Taking the limit $r \rightarrow \infty$ of (4.4) one identifies the asymptotics of this solution as $\mathbb{M}_3 \times S^1$ in a coordinate representation similar to the Rindler slicing of Minkowski space,

$$ds^2 \approx r^2 H_0^2 (-d\tau^2 + \cosh^2(H_0 \tau) dz^2) + dr^2 + R^2 d\theta^2. \quad (4.7)$$

In other words, this geometry asymptotically approaches one of the simple KK compactifications described in (3.6). One can show that the Euclidean version of (4.1)-(4.4) possesses a single negative mode in its spectrum of perturbations, which implies the existence of an instability for these backgrounds.

We can understand the topology of this spacetime by studying its behaviour in the vicinity of the bubble location, at $r \approx 0$ ($B(r) \approx 1$). In this regime equation (4.5) reduces to $r(B) \approx 2H_0^{-1} \sqrt{B-1}$, and the metric has the approximate form,

$$ds^2 \approx -d\tau^2 + \cosh^2(H_0 \tau) dz^2 + dr^2 + r^2 \frac{R^2 H_0^2}{4} d\theta^2, \quad (4.8)$$

which shows that the extra dimension degenerates as one approaches $r = 0$. Moreover, we can see that in order to avoid any conical singularity at $r = 0$, the radius R of the extra dimension must satisfy the relation $R = 2H_0^{-1}$. This means that the transverse directions to the bubble form a kind of smooth cigar geometry that approaches a cylinder of fixed radius at large distances from the tip at $r = 0$ (see figure 1). As a consequence, any loop wrapping the extra dimension can be shrunk to nothing if we take it to the tip of the cigar, so indeed the spacetime is simply connected, which enforces that the fermions be anti-periodic along the extra dimension. Therefore one can consider this as the appropriate geometry of the instanton solution that describes the decay of a non-supersymmetric KK configuration with anti-periodic fermions.

Note that the solution (4.4) depends on a single parameter, H_0 , which gives us the Hubble scale of the two-dimensional de Sitter slice of the spacetime that corresponds to the surface of the bubble, as well as the initial radius of the bubble, $H_0^{-1} = R/2$. Physically, this

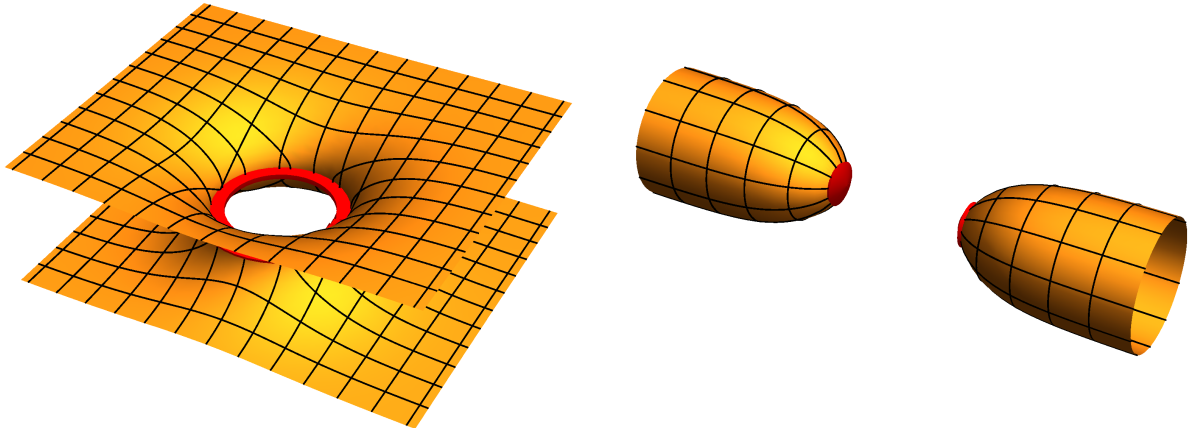


Figure 1. Representation of the bubble of nothing. Left: This figure shows the two large spatial dimensions of the geometry with the apparent vertical separation representing the Kaluza-Klein extra dimension. Right: This image represents the full extra-dimension but only the radial large direction. We use a thick red region in both images to represent the vortex/string present at the interior edge of lower-dimensional spacetime, where the extra dimension degenerates to zero size, as described in Sec. 4.2.

scale parametrizes the deviation of the geometry from Euclidean flat space near the $r \rightarrow 0$ ($B(r) \rightarrow 1$) region. At the same time, it also encodes the information about the size of the compactified extra dimension in the asymptotic limit $r \rightarrow \infty$ ($B(r) \rightarrow \infty$) through the relation $R = 2H_0^{-1}$.

In summary, this solution describes an asymptotically flat 3D spacetime with a compact fourth dimension of radius $R = 2H_0^{-1}$. In the $r \rightarrow 0$ limit, the extra dimension pinches off at a ring, which bounds the excised hole in the two large spatial dimensions, and which expands, eating all of future spacetime. This ring/hole is shown in figure 1, left. The fact that in the full 4D geometry, the bubble is a co-dimension two submanifold is apparent in the right half of figure 1.

4.2 The Bubble of nothing in winding compactifications

As we described in previous sections our model allows for compactifications where the scalar field ϕ winds around the extra dimension. Moreover, the configuration also involves a Wilson line for the vector field. Putting these two features together along the extra-dimension does not augment the energy-momentum tensor. This means that there is no backreaction on the metric due to the presence of these new ingredients, and the solution is still given by the pure KK compactification $\mathbb{M}_3 \times S^1$. However, this background cannot have the same bubble decay channel as in the absence of these winding modes. The reason is that if we imagine the geometry of the vacuum bubble of nothing with the Wilson line wrapping the extra dimension asymptotically, it is clear that since the spacetime is now simply connected, one finds some magnetic field flux in the vicinity of the tip of this cigar geometry (see figure 1). This means

that the analogous bubble of nothing should be dressed with this flux. Here we propose that there is a simple configuration that meets these requirements: situating an Abelian-Higgs vortex at the tip of the bubble of nothing geometry. This vortex has all the appropriate charges to match the asymptotic requirements of our winding background compactification.

Instanton solutions similar to the one discussed here have been presented in the literature in the context of flux compactifications [30–32].¹⁰ On the other hand there is an important difference between those models and the one we study here. Our winding compactification is not a flux compactification. (Indeed we have not induced any potential for the size of the internal dimension.) Even though the presence of the vortex is dictated by the boundary conditions at infinity, unlike a global vortex, there are no long range interactions, so the vortices’ effects are much more localized.

There are several ways to justify our proposed dressing for the instanton. Having identified the necessity of this Abelian-Higgs vortex (cosmic string) on the geometry we would like to convince ourselves that wrapping the string around the ring at the tip of the vacuum bubble of nothing geometry is, in fact, the correct configuration for the string in this background. In order to do that, we will first assume that the Abelian-Higgs vortex is accurately described by the Nambu-Goto (NG) equations of motion, and that it does not distort the background in a significant way. In other words, that one can take the string to be a probe in the background of a bubble of nothing geometry. Taking into account these approximations one can then easily identify a solution of the NG equations of motion of a string sitting at the tip of the bubble of nothing geometry where the circle extra dimension shrinks to zero size. It is perhaps easier to understand this in the Euclidean version of the solution where the string worldsheet is then wrapping the minimal surface sphere at the tip of the cigar geometry. Intuitively it is clear that in this Euclidean geometry there is no other place where this string can go. The Lorentzian continuation of this solution represents a string being stretched by the de Sitter expansion of the bubble of nothing that is eating up the spacetime.

We can now estimate what the effect of this string is on the bubble of nothing geometry, drawing from our experience on cosmic string spacetimes. Assuming a low tension for the string compared to the Planck scale, we can expect that the only effect on the background would be to introduce a deficit angle on the space transverse to the string, similar to what happens for a cosmic string in flat space [36].

One can introduce such a deficit angle on the metric by changing the value of H_0 in (4.4) to make it depend on the tension of the string, and at the same time keeping fixed the radius of the extra dimension R to be what we had before. The last condition ensures that at large distances from the bubble, $r \rightarrow \infty$, the spacetime still asymptotes to a KK geometry $\mathbb{M}_3 \times S^1$ with radius R for the extra dimension, as in (4.7). More specifically we should take

$$2H_0^{-1} = \frac{R}{1 - \frac{\Delta_W}{2\pi}} = \frac{R}{1 - \frac{\mu\kappa^2}{2\pi}} = \frac{R}{1 - 4G\mu}, \quad (4.9)$$

where Δ_W is the *local*¹¹ deficit angle induced by the string of tension μ . (Remember that in our notation $\kappa^2 = M_P^{-2} = 8\pi G$.) We can now repeat the same calculation that we had done before to obtain the form of the metric in the limit $r \rightarrow 0$. After substituting the previous

¹⁰See [33, 34] for a different approach to bubbles of nothing in this field theory context. See also [35] for some discussion of bubbles of nothing in models of flux compactification in the String Theory context.

¹¹A local deficit angle is measured at an infinitesimal distance from the Nambu-Goto string. Farther from the string, Δ_W is the deficit angle removed from Witten’s smooth bubble geometry.

expression in (4.8) we find

$$ds^2 \approx -d\tau^2 + \cosh^2 \left(\frac{2(1-4G\mu)}{R} \tau \right) dz^2 + dr^2 + (1-4G\mu)^2 r^2 d\theta^2. \quad (4.10)$$

This solution has now a conical singularity at $r \rightarrow 0$ signaling the presence of the string of tension μ at that point. Furthermore the circumference of the initial bubble ring, which can be read from the periodicity of $z \in [-\frac{\pi}{H_0}, \frac{\pi}{H_0})$, is now modified by the string, becoming slightly bigger than before: $H_0^{-1} > R/2$. We will refer to the vacuum geometry characterized by $H_0^{-1} \neq R/2$ as a *deformed* bubble of nothing¹².

These solutions describe the most important modifications of the geometry for the bubble of nothing instantons in our model with the winding fields around the extra-dimension. In particular they describe a very interesting property of the model in the limit of $4G\mu \rightarrow 1$. In this *critically deformed* case, one sees that the bubble size for our instanton becomes infinite, $H_0^{-1} \rightarrow \infty$. In other words, the string world-sheet becomes flat and the transverse space to all these solutions corresponds to a cigar-like static geometry. This static, infinite (string wrapped) bubble signals a complete suppression of the tunneling process to nothing, exactly in the same way as what happens in the usual Coleman - de Lucia [3] transition. The Euclidean action in this case diverges, and the decay rate to nothing vanishes much in the same way as it occurs in field theory models without extra dimensions.

It is important to note that these configurations have been found using the thin wall approximation, and it is not clear if the suppression will survive in the full field theory description of our model. In the following sections we will test all these ideas by looking at the smooth numerical solutions of the bubbles of nothing within the Abelian-Higgs model.

5 Bubble of nothing in the Abelian-Higgs model

Previous arguments suggest the existence of solutions describing the decay of a compactified space via the formation of a bubble of nothing where the bubble is *dressed* with a cosmic string. In this section we would like to explore the existence of these solutions in the Abelian-Higgs model where the cosmic string will be represented by a smooth vortex. Our starting point is the action,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - |D_\mu \phi|^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\beta e^2}{2} (\eta^2 - \phi \bar{\phi})^2 \right), \quad (5.1)$$

where we have introduced the deformation parameter β . Note that this action only coincides with the bosonic sector of the supergravity model presented in section 2 for $\beta = 1$, and thus this parameter determines an explicit breaking of supersymmetry for values of $\beta \neq 1$.

We will look for solutions with a generalized bubble of nothing ansatz for the metric, namely solutions of the form (4.4), with the profile functions $B(r)$, $C(r)$ and the parameter H_0 yet to be determined. As in the previous section the induced metric on the bubble wall (and the vortex) is a 2-dimensional de Sitter space with Hubble parameter H_0 , and the initial bubble radius at $\tau = 0$ is given by H_0^{-1} . The limiting behaviour of $C(r)$ for large values of the radial coordinate r fixes the asymptotic size of the compact dimension via the relation $R = \lim_{r \rightarrow \infty} C(r)$.

¹²The thermodynamic properties of the euclidean version of this solution are discussed in [37, 38].

We will consider the configuration for a vortex of unit winding number for the matter fields, which is of the form [36],

$$\phi(r) = f(r)e^{i\theta}, \quad A_\mu = \frac{1}{e}(1 - a(r))\delta_{\mu\theta}. \quad (5.2)$$

In order to simplify the notation, we can redefine fields and lengths by the following rescalings,

$$f \rightarrow \eta f, \quad \tau \rightarrow l_g \tau, \quad r \rightarrow l_g r, \quad z \rightarrow l_g z, \quad C \rightarrow l_g C, \quad H_0 \rightarrow l_g^{-1} H_0, \quad (5.3)$$

where we are using the length scale $l_g \equiv \frac{1}{\eta e}$, corresponding to the vector core thickness. Note that all the coordinates $\{\tau, r, z, \theta\}$ and the parameters R and H_0 are now dimensionless. Using this ansatz we arrive at the matter field equations

$$\frac{(B^2 C f')'}{B^2 C} - \frac{a^2 f}{C^2} + \beta(1 - f^2)f = 0, \quad \frac{C}{B^2} \left(\frac{B^2 a'}{C} \right)' - 2f^2 a = 0. \quad (5.4)$$

Furthermore the t - t component and the θ - θ component of the Einstein's equations read (the r - r component is a constraint)

$$\begin{aligned} \frac{(C B B')'}{B^2 C} &= \gamma \left(\frac{a'^2}{2C^2} - \frac{\beta}{2}(1 - f^2)^2 \right) + \frac{H_0^2}{B^2}, \\ \frac{(B^2 C')'}{B^2 C} &= -\gamma \left(\frac{a'^2}{2C^2} + \frac{2a^2 f^2}{C^2} + \frac{\beta}{2}(1 - f^2)^2 \right), \end{aligned} \quad (5.5)$$

where we have made use of the dimensionless parameter $\gamma = \kappa^2 \eta^2$ introduced earlier which, as we see from these equations, determines the gravitational coupling of the string.

5.1 Compactified vacuum states

Using the ansatz given above it is easy to show that the following configuration solves the equations of motion for arbitrary values of β and γ ,

$$f(r) = 1, \quad a(r) = 0, \quad B(r) = H_0 r, \quad C(r) = R. \quad (5.6)$$

Putting the dimensionful constants back in, we can see that this is nothing more than our original $\mathbb{M}_3 \times S_1$ background given in the same gauge as in eq. (4.7),

$$\phi = \eta e^{i\theta}, \quad A_\mu = e^{-1} \delta_{\mu\theta}, \quad ds^2 = r^2 H_0^2 (-d\tau^2 + \cosh^2(H_0 \tau) dz^2) + dr^2 + R^2 d\theta^2. \quad (5.7)$$

We note that fixing the values of β and γ we completely specify the theory to be considered but we still have the freedom to set the radius of the compactified space, R , to any value. This is just a reflection of the fact that the radius of the extra dimension is a flat direction in the moduli space of the compactified theory. The constant H_0 is also left undetermined in this background, but here it has no physical meaning, it just signals a coordinate freedom associated to the Rindler slicing we are using to parametrize the \mathbb{M}_3 .

5.2 Boundary conditions for the bubble solutions

In order for the spacetime of the bubble of nothing to be everywhere regular, the metric profile functions must satisfy the following conditions on the bubble, that is, at $r = 0$ (see appendix B):

$$C(0) = 0, \quad C'(0) = 1, \quad B'(0) = 0. \quad (5.8)$$

We will also require the gauge condition $B(0) = 1$, in order for the metric to have the form (4.8) in the limit $r \rightarrow 0$. These conditions mean that the geometry near the bubble approaches that of $dS_2 \times \mathbb{R}^2$, where the two factors represent the intrinsic de Sitter geometry on the bubble and the smooth end of the compact dimension, respectively. In other words, for $r \approx 0$ we have

$$ds^2 \approx -d\tau^2 + \cosh^2(H_0 \tau) dz^2 + dr^2 + r^2 d\theta^2. \quad (5.9)$$

Note that, since $z \in [-\frac{\pi}{H_0}, \frac{\pi}{H_0})$, the limit $H_0 \rightarrow 0$ of this geometry is locally that of 4-dimensional Minkowski space.

For the matter field configuration to be regular at the bubble location we must also require that

$$f(0) = 0, \quad a(0) = 1. \quad (5.10)$$

Solutions representing a bubble of nothing in a winding compactification should approach a KK vacuum of the form (5.7) asymptotically, therefore the profile functions must have the following asymptotic behaviour for $r \rightarrow \infty$:

$$\lim_{r \rightarrow \infty} f(r) = 1, \quad \lim_{r \rightarrow \infty} a(r) = 0, \quad \lim_{r \rightarrow \infty} C(r) = R. \quad (5.11)$$

Note that, a priori, the parameter H_0 appears to be unfixed by the boundary conditions. However once we demand the profile functions to meet all the boundary conditions specified above, there is a unique value of H_0 which is compatible with them and the fields equations.

In summary, we will obtain the parameter H_0 , together with the form of the profile functions of the metric and matter fields as a result of numerically solving eqs. (5.4) and (5.5), subject to the conditions (5.8), (5.10) and (5.11).

6 Comparing numerical results with the thin wall approximation

We have shown in previous sections that one can obtain a supersymmetric compactification by specifying the condition,

$$\beta = 1, \quad \gamma = 1. \quad (6.1)$$

In the following we will consider different values of these parameters and their approach to the supersymmetric limit. In this section we will consider the regime of parameter space where the vortex size is much smaller than the compactification radius R and the initial bubble size H_0^{-1} . This is the situation where we expect the thin wall approximation discussed in section 4.2 to be an accurate description.

6.1 Non-supersymmetric compactification: the $\gamma \neq 1$ case

We start our investigation by looking at solutions where $\beta = 1$ and the gravitational coupling of the vortex is small, $\gamma \ll 1$. It is clear that in this case, the compactification would break supersymmetry spontaneously since the condition in eq. (3.10) will not be satisfied.

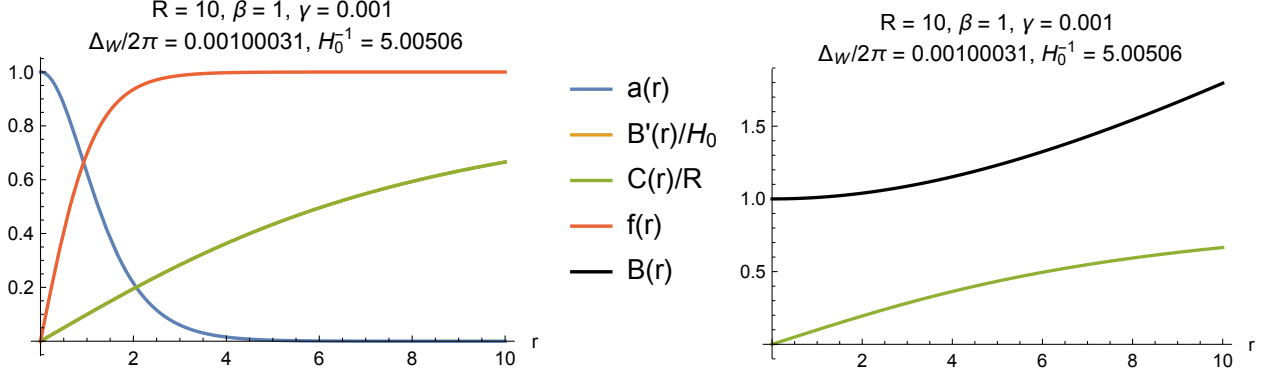


Figure 2. Bubble of nothing in the Abelian-Higgs model with $\beta = 1$, $\gamma = 10^{-3}$ and asymptotic radius of the compact dimension $R = 10$. Note that since $R \gg 1$, the vortex is much thinner than the background curvature, and so we can compare the bubble deficit angle $\Delta_W \approx 2\pi 10^{-3}$ with the flat space deficit angle $\Delta = 2\pi 10^{-3}$ in eq. (6.2). This is a very light vortex with tension $4G\mu = 10^{-3}$, so backreaction is almost negligible. In the left panel, the matter field profiles are shown, together with the metric fields $B'(r)/H_0$ and $C(r)/R$. These last two are on top of each other, inferring that we are in a bubble of nothing configuration, as given by (4.6). This can be corroborated by the profiles of $B(r)$ and $C(r)$ in the right panel.

Keeping $\beta = 1$, the vortex solution is special in the sense that the scalar and magnetic cores are of the same size, which is unity in our current units¹³ [36]. On the other hand, we still have the freedom to fix the size of the compact dimension, so it makes sense to start our investigation in the regime where there is a clear separation of scales between the size of the vortex and the size of the extra dimension, $R \gg 1$. It is in these cases that we expect a bubble of nothing solution that is very similar to Witten’s pure gravity solution. Furthermore, small variations from this solution should be well captured in this regime of parameters by our analysis within the thin wall approximation.

In figure 2 we present a numerical solution that corresponds to a bubble of nothing in such a regime, where we have fixed $\gamma = 10^{-3}$ and $R = 10$. We relegate to Appendix B the detailed explanations of the numerical procedure we use in order to find these solutions. Using those techniques and given the values of the parameters (β, γ, R) , we are able to compute both the values of the initial size of the bubble H_0^{-1} , and the vortex induced deficit angle Δ_W that one can infer from the asymptotic vacuum solution.

It can be shown that the effect of the string vortex on the bubble of nothing is negligible and the profile functions $B(r)$ and $C(r)$ resemble almost exactly the form given by Witten’s bubble configuration (4.4) and (4.5). The relation (4.6) is satisfied everywhere, as it can be checked in figure 2, where the line representing $B'(r)/H_0$ remains hidden by the one associated to $C(r)/R$. As mentioned before, this property is characteristic of the pure vacuum bubbles of nothing. Another way to quantify this is by looking at the ratio between the values of R and H_0^{-1} , which in this case is very close to 2 as in the original bubble of nothing. In the right panel of figure 2 we have displayed the asymptotic regime of the profile functions $B(r)$ and $C(r)$, showing that the spacetime geometry approaches $\mathbb{M}_3 \times S^1$ in the limit $r \rightarrow \infty$.

The most notable difference with (4.4) and (4.5) is given for the asymptotic presence of a deficit angle on this solution, which in our numerical solution is $\Delta_W \approx 2\pi 10^{-3}$. We can

¹³In flat space, the value of β determines if we are in the type-I ($\beta < 1$) or type-II regime ($\beta > 1$). $\beta = 1$ is the Bogomolnyi limit. For $\beta = 1$ the vortex core radius is given by l_g without the rescalings in eq. (5.3).

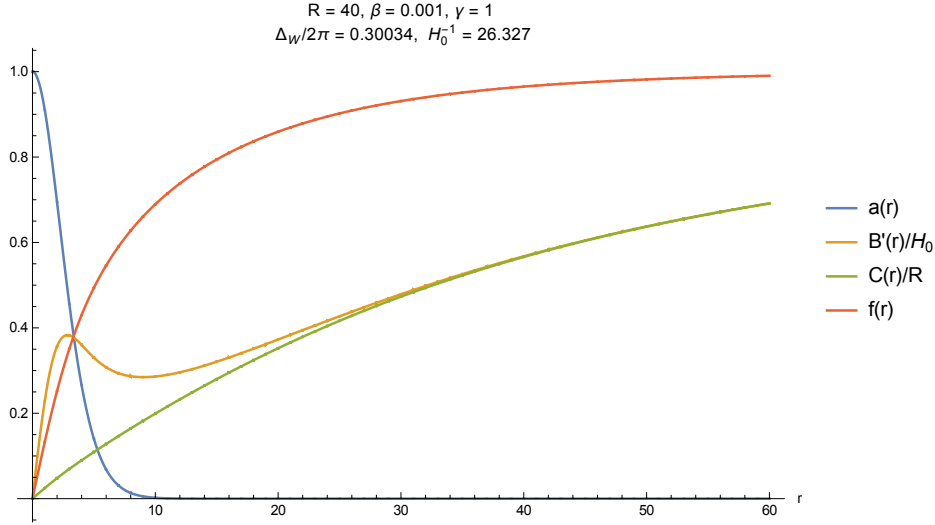


Figure 3. Bubble of nothing in the Abelian-Higgs model with explicit breaking of supersymmetry, $\beta = 0.001$, strong gravitational coupling $\gamma = 1$, and large asymptotic radius of the compact direction $R = 40$. Note that the relation (4.6) is satisfied far from the core.

now compare this value with the deficit angle for the vortex in an asymptotic conical space for the same values of the parameters β, γ . In our case, we can use the result for the case for a vortex of a single unit of flux [36],

$$\Delta(\gamma)|_{\beta=1} = 8\pi G \quad \mu(\gamma)|_{\beta=1} = 2\pi\gamma = 2\pi 10^{-3}. \quad (6.2)$$

We see that the result obtained from the numerical calculation agrees perfectly with the analytic results described earlier. Furthermore, the profile of the matter fields, $f(r)$ and $a(r)$, are not very much affected by the existence of the bubble. This is not surprising, since the radius of the bubble $H_0^{-1} \approx R/2 = 5$ is large compared to the size of the defect.

6.2 Explicit supersymmetry breaking: the $\beta \neq 1$ case

We can also explore the regime where we set $\gamma = 1$ and vary β . In this case, the original theory breaks supersymmetry explicitly, so we expect the existence of bubbles of nothing in this case as well. The thickness of the scalar vortex gets bigger when decreasing the value of $\beta < 1$, so in order to check the validity of the thin wall approximation we should also consider large values of the extra dimensional space and integrate the equations to larger distances from the core.

We show in figure 3 a solution for $\beta = 0.001$, and $R = 40$. This solution agrees well with the thin wall approximation of a conical defect produced by the analogue vortex in an asymptotically locally flat spacetime that is placed on a bubble of nothing geometry. For instance we see that far from the vortex core ($r \gg 1$) the metric profile functions approach those of the standard bubble of nothing (4.4) and (4.5), and satisfy the characteristic relation (4.6).

Although the string tension decreases for smaller values of the parameter β the dependence is logarithmic [36], which means that it is dominated by the large value of γ in our case. This explains why there is still a quite important backreaction on the value of $H_0^{-1} \approx 26 > R/2 = 20$ compared to the pure bubble of nothing geometry in our example.

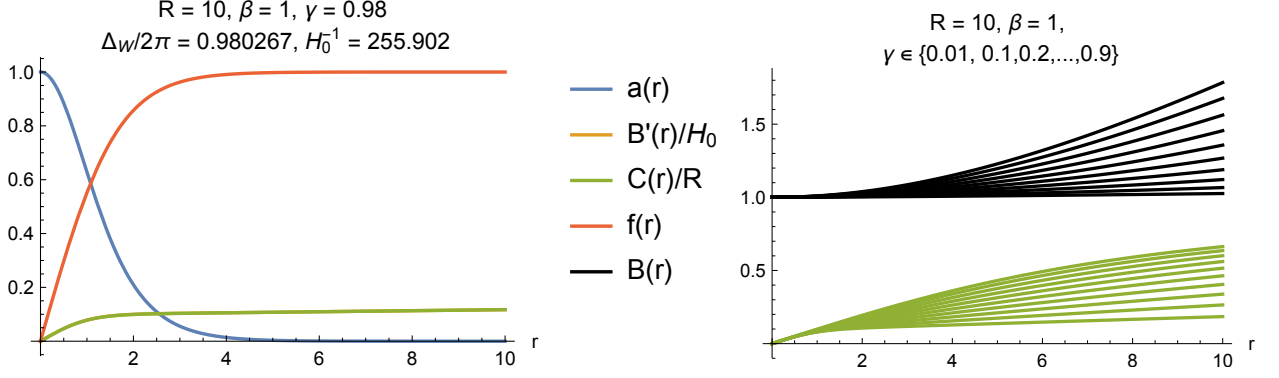


Figure 4. In the left panel we plot a bubble of nothing configuration near the supersymmetric limit ($\gamma = 0.98$). Note that although the matter fields rapidly approach their asymptotic (vacuum) values, in the region plotted the extra dimension has size $C(r) \approx 1$, and it only very slowly approaches its asymptotic value of 10. The right plot shows a sequence of bubble of nothing geometries for increasing values of γ , fixed $\beta = 1$, and $R = 10$. From top to bottom $\gamma = \{0.01, 0.1, 0.2, 0.3, \dots, 0.9\}$, showing that for higher values of γ there is a larger region ($1 \lesssim r \lesssim H_0^{-1}$) where the metric profile functions $B \approx 1$ and interestingly, $C \approx 1$, regardless of the value of R .

Note also that the equation (6.2) for the deficit angle of the vortex in an asymptotically conical spacetime is only valid for $\beta = 1$, and therefore it cannot be used in the present case to predict the approximate value of Δ_W . Nevertheless, we have checked that the obtained Δ_W agrees well with the analogue value in a conical spacetime for the same parameters.

These solutions and their agreement with the thin wall approximation described in section 4.2 validates our numerical techniques and demonstrates explicitly the existence of these decay channels in models with broken supersymmetry.

6.3 Supersymmetric limit: Approaching the critical bubble

Taking as our initial condition the solutions found previously, we would like to find what happens as one approaches the supersymmetric compactification limit where $\beta = 1$ and $\gamma = 1$. As in the previous subsections, we will restrict our attention to the case where the asymptotic compactification radius is large compared to the vortex width, $R \gg 1$, so that the predictions of the thin wall approximation are applicable. Nevertheless, due to the large value of the gravitational coupling, γ , the backreaction of the vortex on the geometry is expected to be large, and thus to induce significant deviations from Witten's bubble of nothing given by equations (4.4) and (4.5). We show a sequence of the numerical solutions in the right panel of figure 4 with the parameter β fixed to unity, the asymptotic size of the compact dimension set to $R = 10$, and γ varying in the range $[0.01, 0.9]$.

Moving slowly in the parameter space towards this supersymmetric limit, we see the appearance of two different vacuum regions. On the one hand, for $0 \leq r \ll H_0^{-1}$, the geometry of the space adjacent to the vortex becomes increasingly similar to the background of a string in an asymptotically conical spacetime with critical tension, i.e. with deficit angle $\Delta \approx 2\pi$ [36]. Outside the vortex core, $1 \ll r \ll H_0^{-1}$, this vacuum geometry resembles a static cylinder where the radius of the extra dimension is constant and equal to the *vortex size*, which in our units is $C(r) = 1$. This region of space is displayed in figure 4. At large distances from the vortex core, $r \gtrsim H_0^{-1}$, the spacetime begins to resemble the pure compactification, and matches the boundary condition $\lim_{r \rightarrow \infty} C(r) = R = 10$. It is instructive to view the

different behaviors over a large range of r , and so we will plot the profile functions over several decades in r in later sections (e.g., right panel of figure 5), showing how the solutions interpolate between these two regimes.

As one approaches the supersymmetric case, ($\gamma = 1, \beta = 1$), the value of H_0 decreases quickly, becoming zero in that limit. Recall that H_0 is the Hubble scale of the induced de Sitter geometry on the bubble surface. The vanishing of this parameter thus indicates the arrival at a flat bubble geometry. The values of H_0 obtained numerically are in very good agreement with the predictions from the thin wall approximation, i.e., with equations (4.9) and (6.2). Note that the circumference of the bubble at the moment of nucleation is $2\pi H_0^{-1}$.

In summary, in the supersymmetric limit the bubble becomes flat and infinitely large. Moreover, as one approaches this limit, the geometry of the transverse directions ever more slowly interpolates between two regimes:

- $1 \lesssim r \ll H_0^{-1}$: outside (but near) the vortex core, the geometry resembles a cylinder of approximately constant radius $C(r) \approx 1$. This region becomes infinitely large in the supersymmetric limit where $H_0 \rightarrow 0$.
- $H_0^{-1} \ll r$: far from the vortex core, the compact dimension approaches the asymptotic radius $C(r) \approx R$. In the supersymmetric limit ($H_0 \rightarrow 0$), this asymptotic regime occurs at an infinite distance from the bubble, assuming $R \neq 1$.

We show in the left panel of figure 4 the profiles obtained close to the limiting case ($\gamma = 0.98$). In this example the initial bubble size is given by $H_0^{-1} \approx 256$, which still in good agreement with the prediction obtained from the thin wall approximation

$$H_0^{-1}|_{\text{tw}} = \frac{R}{2(1 - 4G\mu)} \approx 250, \quad (6.3)$$

where we have used (4.9) and (6.2). This is important for the calculation of the decay rate of the compactified spacetime, since an infinite bubble would give rise to an infinite action for the instanton and therefore a total suppression of the tunneling transition. This is exactly what happens in the usual Coleman-De Luccia suppression mechanism.

Although not plotted here, we have repeated the numerical calculations following different paths towards the supersymmetric limit in the (β, γ) parameter space while keeping fixed the value of R , and in it is interesting to note that the behaviour we just discussed is independent of the path.

7 Numerical results away from the thin wall regime

As we noted earlier, all possible values of the asymptotic KK radius R are allowed for each point in the parameter space (β, γ) . We expect that there will be instanton solutions representing the decay to a bubble of nothing for all these values. We have argued that these instantons should involve the presence of a vortex in their geometry that should fit inside the bubble of nothing solution with the correct asymptotics. In the previous sections we have shown explicitly that this is possible in our smooth Abelian-Higgs model when there is a clear separation of scales between the compactification size and the vortex core size $R \gg 1$.

Here we would like to numerically explore what happens when we are not in the above regime, in other words, when we are well outside of the region of validity of the thin wall approximation. This is not just a technical curiosity, it is an important point for the conclusions

of our paper. We are arguing that the supersymmetric limit of our compactification is protected from the bubble of nothing decay dynamically by the Coleman-De Luccia suppression mechanism. If this is the case, it should be the same for any value of the compactification radius R , not only for the situations that are easily described by the thin wall approximation. We therefore extend our investigations to some of the cases where one can only find the solution by performing the numerical integration of the equations of motion.

7.1 Small compactification radius, $R \lesssim 1$

The situation seems more problematic in cases where the compactified space is smaller than the vortex core. It would seem difficult to find the bubble of nothing instanton of the kind that we have been discussing, since there seems to be no space for the vortex to fit in this geometry.

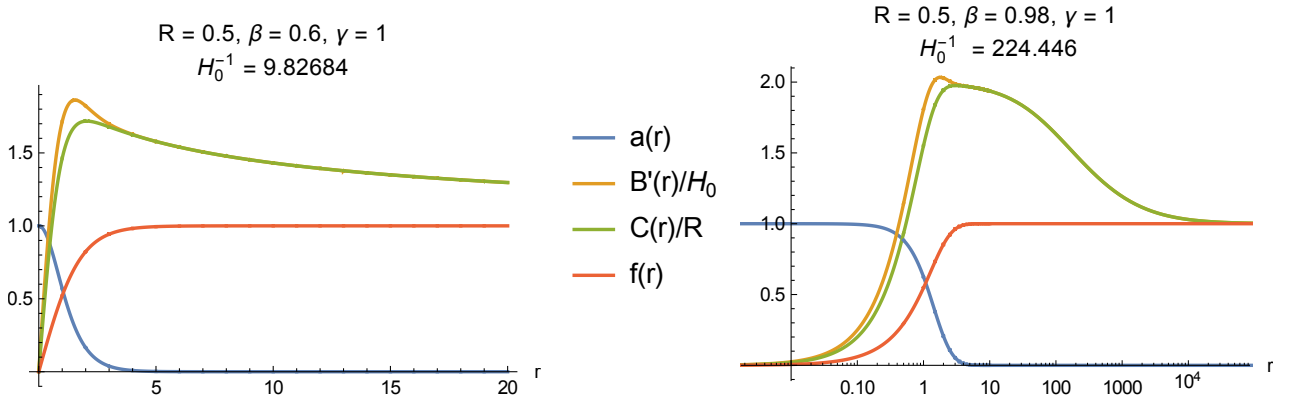


Figure 5. A bubble of nothing for a winding compactification with KK radius $R = 0.5$ and gravitational coupling $\gamma = 1$. The left panel shows the solution for $\beta = 0.6$. The right panel shows the ($\beta = 0.98$) bubble of nothing, close to the supersymmetric case. In this panel the radial distance is arranged on a logarithmic scale so that the figure shows the two regimes: inside/near the core, and asymptotically the compactification boundary conditions.

In the left panel of figure 5, we present an example of such an instanton for the case of $R = 0.5$, with supersymmetry breaking parameters $\gamma = 1$ and $\beta = 0.6$. We see that the solution does exist, but its geometry is quite different from the previous cases. Close to the vortex, the extra dimension is larger than its asymptotic value. This is due to the presence of the vortex matter fields that force the extra-dimensional volume to be large enough to hold the vortex. Once the matter fields are settled near their vacuum values $f(r) = 1$ and $a(r) = 0$, the geometry relaxes (possibly very slowly) to the one imposed by the boundary condition at infinity. In particular, we see that the metric profile functions satisfy the characteristic relation (4.6). As we discussed above, the regions where this relation is satisfied signal that the spacetime metric is an approximate solution of the vacuum Einstein's equations. The interesting point about this configuration is that it settles to a vacuum solution which corresponds to a Schwarzschild-like solution of the type given by eq. (4.1) but with a negative mass term. Indeed, when the metric is written in the gauge (4.4) this means that the metric profile function $C(r)$ has the following asymptotic behaviour for $r \gg 1$ ($B(r) \gg 1$):

$$B'(r) \approx H_0^{-1} > 0, \quad C(r) \approx R \sqrt{1 + B(r)^{-1}}, \quad (7.1)$$

which can be obtained proceeding as in section 4.1, but setting $\rho_0 < 0$. This explains how the size of the extra dimension, $C(r)$, can be a decreasing function of the distance from the core, as figure 5 shows.

A vacuum solution with this behavior cannot exist on its own, since this would lead to a naked singularity, not a smooth bubble of nothing geometry. The reason is that a negative mass Schwarzschild solution does not have a horizon, so both the Lorentzian geometry as well as its analytic continuation would be singular. It is only due to the presence of the vortex that one can cap the geometry, replacing the singularity by the smooth vortex.¹⁴ In this sense, these solutions are clearly not a deformation of the usual vacuum bubble of nothing geometry.

Taking the supersymmetric limit of these solutions, we arrive at the same conclusion as in the previous section. As one approaches the $\beta = 1, \gamma = 1$ limit, the size of the bubble, H_0^{-1} diverges, signaling again the suppression of the decay. We show in the right panel of figure 5 an example of such behaviour for $R = 0.5$, and parameters $(\gamma = 1, \beta = 0.98)$. In this case the value we obtain for the Hubble parameter is $H_0 \approx 4 \times 10^{-3}$. We have plotted the solution on a logarithmic axis to see clearly the two regions in the solution (which we described in section 6.3), the vortex cylinder close to the tip, $1 \lesssim r \ll H_0^{-1}$, and the compactification geometry at large distances, $H_0^{-1} \lesssim r$. Outside the vortex, this case corresponds to an analytically continued Schwarzschild solution with a negative mass term. In particular, we see that for $1 \lesssim r \lesssim 200$ the radius of the compact dimension is given by the vortex size $C(r) \approx 1$, or equivalently $C(r)/R \approx 2$, and far away from the core, $r \gg 200$, it has decreased toward the asymptotic value $C(r) \rightarrow R$.

7.2 Intermediate regime, $R \sim O(1)$

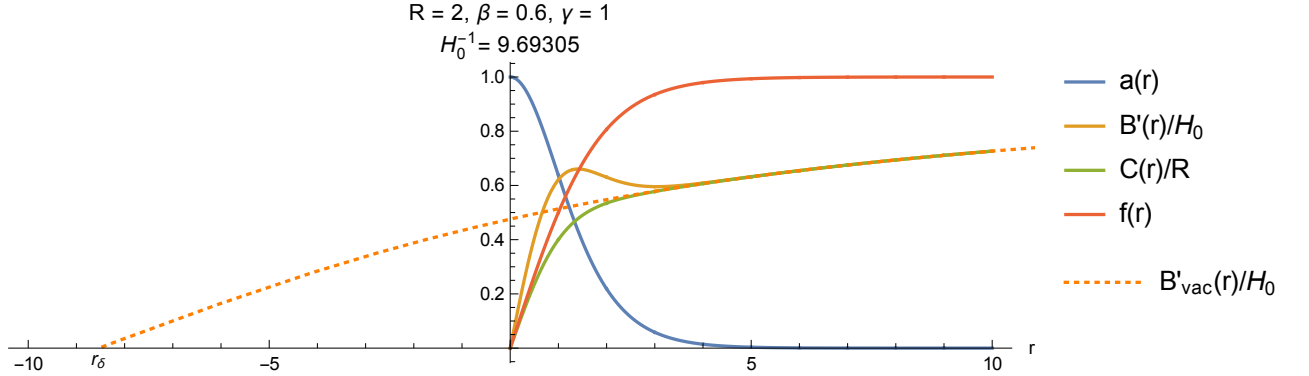


Figure 6. A bubble (solid lines) superimposed with the corresponding asymptotic vacuum solution $B'_{\text{vac}}(r) = H_0 C_{\text{vac}}(r)/R$ (dotted). Numerically we also find $H_0 = 0.103$ even though the asymptotic vacuum solution would extrapolate to a smaller (Nambu-Goto wrapped) bubble with $H_\delta = 0.132$. We can characterize this “thick-vortex” effect as the ratio $H_\delta/H_0 = 1.29$.

We can also explore numerically the solutions that interpolate between the extreme cases discussed earlier, i.e., cases with very small and very large compactification radius. In figure 6 we show a solution with $R = 2$. We have also represented, with a dashed line, the deformed bubble solution of the vacuum Einstein’s equations given by (4.4) and (4.5), which

¹⁴It would be interesting to investigate these new configurations as regular Euclidean black hole solutions with negative mass terms.

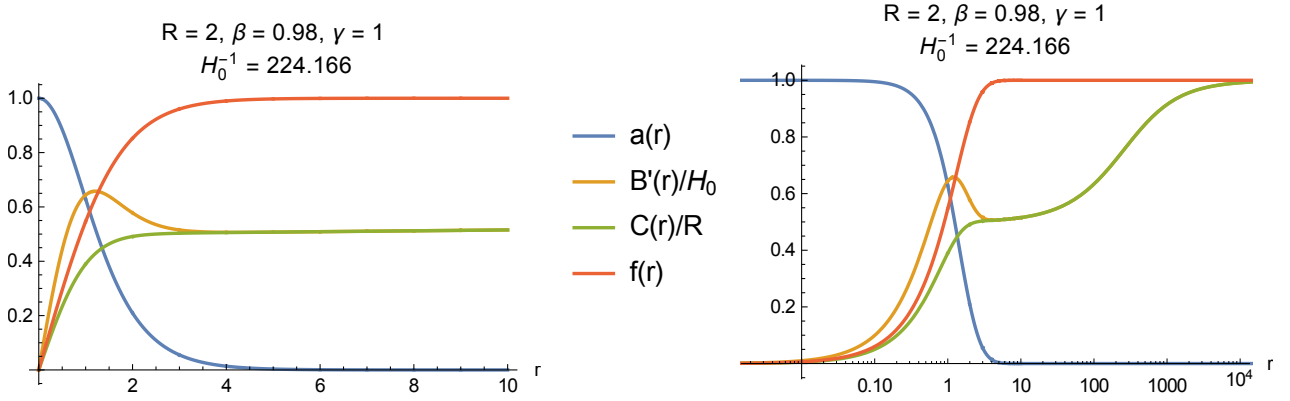


Figure 7. Solutions for $\gamma = 1$ and $R = 2$ close to the supersymmetric limit, $\beta = 0.98$. Note that the initial bubble radius $H_0^{-1} \gg 1$ is very large. The behaviour of the profile functions near the core of the vortex, $r \lesssim H_0^{-1}$, are shown in the left plot. In the right plot the radial coordinate is represented in logarithmic scale in order to display the behaviour far from the vortex core $r \gg H_0^{-1}$.

matches the same asymptotic behaviour of the fully numerical solution. As we discussed in section 4, such a vacuum solution can be completely characterized by the boundary condition $R = 2$, as well as the deficit angle $\Delta_W = 5.45$, and satisfies the relation (4.6) *everywhere*. Such solutions require a delta-function (Nambu-Goto) source to induce the deficit angle deformation of Witten's bubble, Δ_W . It should be located at a point $r = r_\delta < 0$ to correctly match the profile functions for $r \rightarrow \infty$. Instead, in the fully numerical Abelian-Higgs solution, the presence of matter causes a sudden drop in both C and B' , with the bubble appearing at $r = 0$, rather than the extrapolated value $r = r_\delta$.

Similarly to what we did before, we show in figure 7 the solution near the supersymmetric limit, $\beta \rightarrow 1$. The behaviour is similar to the other cases, and in particular we observe that H_0 becomes arbitrarily small, implying that the suppression persists for all values of the asymptotic compactification radius R . This intermediate regime allows us to distinctively see both regions of the deformed bubble solution, the vortex core which resembles a static vortex solution and the large r geometry that matches the pure compactification. In figure 7, we display the spacetime region near the vortex core $r \lesssim H_0^{-1}$ on the left panel, and the transition between the vortex core region and the asymptotic geometry for $r \gtrsim H_0^{-1}$ is shown on the right. As usual, $C'(r) \approx 1$ while $1 \lesssim r \lesssim H_0^{-1}$, after which it approaches its asymptotic value R .

7.3 Limiting case $R = 1$, and the half-BPS solution

Finally we discuss the special case where the extra dimension is such that the asymptotic radius of the compact dimension is $R = 1$. This is a particularly interesting case, since in our conventions, this corresponds to the natural size of the vortex in the supersymmetric limit of the theory, namely when $\beta = 1$ and $\gamma = 1$.

We cannot rely on the thin wall approximation in this case either since there is no real separation between the size of the vortex and the compact space volume, specially if we take the $\beta < 1$ case where the vortex core would be even bigger, as follows from the experience in asymptotically conical spacetimes. We show an example of these solutions with $\beta = 0.6$ in figure 8, which proves the existence of bubble of nothing configurations in this regime. Nevertheless it is interesting to note that taking the supersymmetric limit one arrives to practically

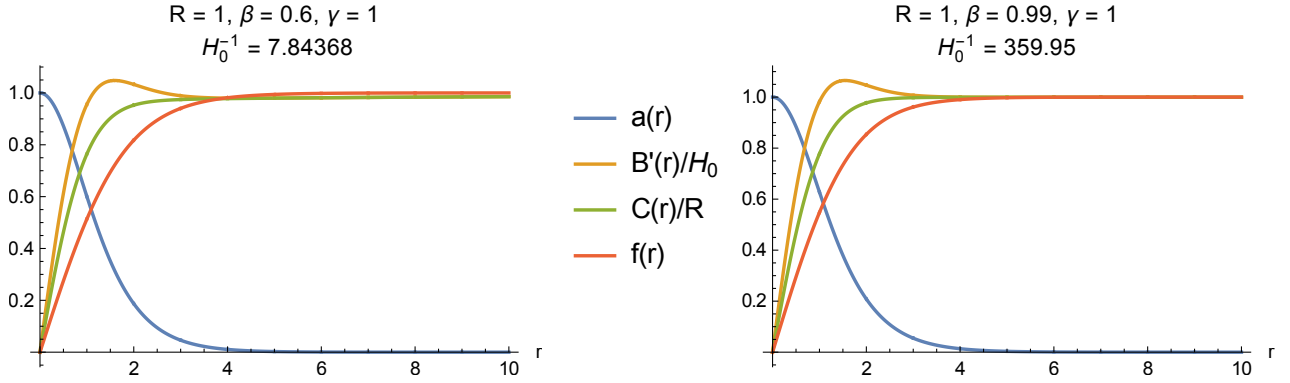


Figure 8. On the left plot we show a bubble with gravitational coupling $\gamma = 1$ and KK radius $R = 1$ away from the supersymmetric state, $\beta = 0.6$. Note the important deviation from the thin wall approximation. The right plot shows a bubble with $\gamma = 1$ and $R = 1$ near the supersymmetric limit, $\beta = 0.99$. The radius of the bubble H_0^{-1} becomes very large in this case. All indications are that $H_0 \rightarrow 0$ as $\beta \rightarrow 1$ from below.

the same conclusions as in the previous cases, as shown in the right panel of figure 8: the bubble becomes flat and infinite, signaling an exponential suppression of the decay rate.

Let us now discuss more in detail the limiting solution we obtain for $R = 1$, as we approach the supersymmetric case ($\gamma = 1, \beta = 1$). In previous sections we have shown that in this limit the spacetime metric displays two different regimes. On the one hand, near the string core, $1 \lesssim r \lesssim H_0^{-1}$, the geometry is dominated by the vortex configuration and $C(r) \approx 1$. On the other hand, at large distances from the core, $r \gg H_0^{-1}$, the metric approaches the asymptotic KK configuration with radius $C(r) = R$. Our numerical calculations also indicate that H_0 vanishes in the supersymmetric limit, which implies that the first regime becomes infinite. Interestingly, in the case $R = 1$ the near vortex configuration already meets the asymptotic boundary condition $C(r) = R = 1$, and therefore it would seem that there is no sense in which two regimes are present. As we shall argue in the following, this is precisely the case at hand. Actually, the resulting configuration is a half-BPS vortex solution that is known to exist in our model [9, 10].

Following the evidence obtained by our previous numerical calculations we will set H_0 to zero in order to study the supersymmetric limit. Then, the generalized ansatz for the metric (4.4) reduces to

$$ds^2 = B^2(r) (-d\tau^2 + dz^2) + dr^2 + C^2(r) d\theta^2. \quad (7.2)$$

Note that the coordinate z now takes values in the range $(-\infty, \infty)$, implying that the bubble radius is infinite, and therefore the string wrapping it is also infinitely long and flat. If we set $H_0 = 0$ in the system of equations (5.4) and (5.5), it can be shown that they admit a first integral (see for example [39]), leading to a new system of first order differential equations called the BPS equations,¹⁵

$$f' - faC^{-1} = 0, \quad a' - C(f^2 - 1) = 0, \quad C' - 1 + \gamma(1 + a(f^2 - 1)) = 0, \quad (7.3)$$

¹⁵Although we are discussing the supersymmetric limit, we leave here the parameter γ explicit for later convenience.

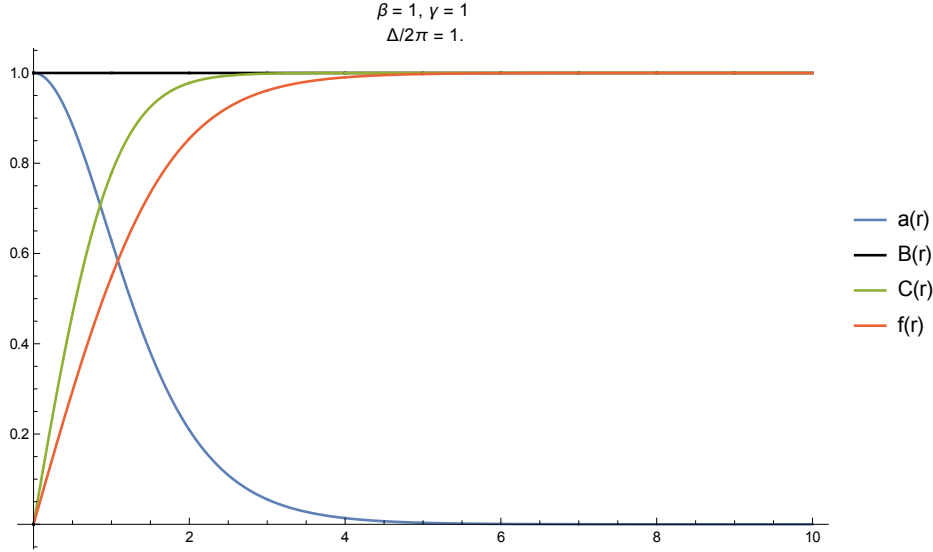


Figure 9. Solution in the supersymmetric limit $\gamma = 1$, $\beta = 1$, with asymptotic compactification radius $R = 1$.

while the profile function $B(r)$ can be consistently set to a constant, $B(r) = 1$. Furthermore, the boundary conditions that impose regularity at the core as well as finite energy (per unit length of the string) reduce to

$$C(0) = 0, \quad f(0) = 0, \quad a(0) = 1, \quad f(r \rightarrow \infty) = 1, \quad a(r \rightarrow \infty) = 0. \quad (7.4)$$

Field configurations satisfying the BPS equations and these boundary conditions can be shown to leave unbroken half of the supersymmetries. More precisely, the unbroken supersymmetries are those generated by a parameter ϵ of the form

$$\epsilon_L(\theta) = e^{-\frac{1}{2}i\theta} \epsilon_{0L}, \quad (7.5)$$

where ϵ_0 satisfies the projector condition $\gamma^{12}\epsilon_0 = -i\gamma^5\epsilon_0$ [9, 10]. Far away from the core, $r \rightarrow \infty$, the BPS equation for the metric profile function $C(r)$ becomes

$$C' \approx 1 - \gamma, \quad (7.6)$$

therefore, if we want this spacetime to behave asymptotically as a compactified state we must require that $C'(r) \rightarrow 0$ for large r , and thus we need the parameters of our theory to obey the constraint $\gamma = 1$. In this case, for $r \rightarrow \infty$, the solution approaches the vacuum compactification discussed earlier and given eq. (3.8), where supersymmetry is fully restored since the parameter γ satisfies the constraint (3.10). (See [10].) Furthermore, the θ -dependence of the supersymmetric parameter (7.5) is consistent with the boundary conditions of the fermions, which should be anti-periodic, as the spacetime is simply connected.

In figure 9 we show a numerical solution of the previous system of equations and boundary conditions. Note that the radius of the compactification rapidly approaches the asymptotic value $C(r) = R = 1$, and therefore this configuration is also a solution of the full set of equations of motion and boundary conditions imposed in section 5.2, which we used previously to obtain the bubble configurations. It is interesting to note that the field profiles in

this solution are identical to the ones we have found earlier for other values of R near the supersymmetric limit (see e.g. figure 4). This is of course possible because those solutions also have a very small H_0 in that limit, so the bubble becomes effectively flat, and the outer region $r \gtrsim H_0^{-1}$ is at a very large distance from the core. Indeed, as we anticipated at the beginning of this section, in the solution of (7.3) and (7.4) represented in figure 9, the outer regime is totally absent.

The spacetime in the (r, θ) directions resembles a cigar type geometry, and it is pretty close to that of a cylinder with a spherical cap attached to it on its end [10, 39]. Taking into account this description of the solution, one can interpret the half-BPS cosmic string solution as an interpolation between two different vacua. The compactified vacuum $\mathbb{M}_3 \times S^1$ given by eq. (3.8) and the pure magnetic spherical compactification $\mathbb{M}_2 \times S^2$ at the vortex core, which can also be shown to be a solution in our model [10, 39]. In this view, the solution is very similar to the static supersymmetric domain walls that interpolate between supersymmetric vacua in supergravity theories [5]. The presence of these half-BPS solutions in those models signals the suppression of a possible decay between such vacua, which is precisely the behaviour we encounter in our case. In our model, the decay to the bubble of nothing is suppressed in this case when $R = 1$ by the appearance of a half-BPS cosmic string solution that prevents the decay from happening. Other possible initial supersymmetric configurations are also protected by a similar object, although in these cases the solution is not half-BPS due to different asymptotic boundary conditions.

8 Conclusions

Several years ago, Witten showed that compactified higher dimensional theories are susceptible to decay via the formation of a bubble of nothing. This happens by the spontaneous nucleation of a bubble where the extra-dimension pinches off and disappears. It is generally believed that supersymmetric compactifications would be stable with respect to this decay channel due to the necessity of periodic fermions around the extra dimension (circle of compactification). This periodicity is incompatible with the bubble of nothing cigar geometry, which imposes antiperiodic fermions in the asymptotic region, the region that approaches the compactification vacuum state.

On the other hand, this KK vacuum solution is not the only possible compactification on a circle that preserves supersymmetry. We have shown in this paper an explicit example of a supersymmetric compactification that allows anti-periodic fermions due to the presence of extra matter fields winding the extra dimension. It would therefore seem possible for these states to decay via an instanton similar to the one in the pure Witten bubble. Here we have investigated this possibility and concluded that one can indeed find such instantons in those models. The new ingredients in this vacuum solutions makes it necessary for a vortex to be placed on the instanton geometry in order to reconcile the asymptotic boundary conditions with the pinching off of the extra dimension. This vortex string can be chosen to only mildly deform the geometry when the parameters of the theory are far from the supersymmetric case, so one can expect these states to be unstable to the formation of the bubble of nothing. However, in the limit where the compactification is supersymmetric, the solution is such that the bubble becomes infinite and flat, signaling the suppression of the instability. This is exactly the same kind of behaviour one encounters in field theory models and shows that, at

least in this case, the suppression of the decay is not due to any topological obstruction or superselection rule, but rather it has a dynamical origin.

We have investigated the behaviour of the bubbles within a simple 4D model, where we can use the thin wall (Nambu-Goto) approximation to estimate the effect of the vortex on the spacetime geometry. Furthermore, we have done a thorough numerical exploration of this model for different initial compactification scenarios and parameters, and we have concluded that this effect is realized in all the cases, even in cases where the thin wall approximation would not be appropriate. This gives us confidence to speculate that this mechanism is generic. It would be interesting to investigate the presence of this suppression mechanism in higher dimensional models of flux compactification in field theory as well as String Theory.

We conjecture that this mechanism stabilizes any supersymmetric compactification that is not prevented from decay by topological obstructions such as spin structure.

Acknowledgments

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A Quantization of the Fayet-Iliopoulos term

In the $\mathcal{N} = 1$ locally supersymmetric version of the Abelian-Higgs model, the parameter η^2 , i.e. the vacuum expectation value of the Higgs field ϕ , is the so called Fayet-Iliopoulos (FI) term. This parameter also determines the charge of the gravitino under the local $U(1)$ gauge transformations and, as a consequence, it satisfies a quantization condition which takes the form [40, 41]

$$\eta^2 \kappa^2 = 2p, \quad \text{where } p \in \mathbb{Z}. \quad (\text{A.1})$$

For simplicity in the main text we have neglected this condition and treated the FI-term as a continuous parameter, but it is straightforward to show that our conclusions are not affected when the quantization is taken into account. In particular we will now show that this model admits a supersymmetric compactification of Minkowski space to $\mathbb{M}_3 \times S^1$ compatible with anti-periodic boundary conditions for the fermions on the S^1 . For this purpose we need to consider a $\mathcal{N} = 1$ locally supersymmetric Abelian-Higgs model with slightly more general couplings than the one defined by (2.1). The bosonic sector of the theory we will discuss now is given by

$$S_{\text{bos}} = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - D_\mu \bar{\phi} D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{e^2}{2} (\eta^2 - q\phi\bar{\phi})^2 \right], \quad (\text{A.2})$$

where the gauge covariant derivative is defined by $D_\mu \phi = (\partial_\mu - iqeA_\mu)\phi$, and the integer $q \in \mathbb{Z}$. Note that, in contrast with the model given by (2.1), the charge of the Higgs is an arbitrary integer multiple q of the gauge coupling e . As we shall now see, introducing this new parameter is essential for the construction of the supersymmetric compactification with anti-periodic fermions. Note also that this model only admits zero-energy vacuum solutions,

such as the spontaneous compactification to $\mathbb{M}_3 \times S_1$ that we wish to discuss, provided the parameters satisfy

$$\text{sign}(p) = \text{sign}(q), \quad (\text{A.3})$$

which we shall assume in the following discussion. The line element corresponding to the spontaneous compactification $\mathbb{M}_4 \rightarrow \mathbb{M}_3 \times S_1$ has the form

$$ds^2 = -dt^2 + dz^2 + dr^2 + R^2 d\theta^2, \quad (\text{A.4})$$

where R is the radius of the compact S^1 direction. In addition we will impose anti-periodic boundary conditions for the all the fermions, χ , λ and ψ_μ , along the compact S^1 direction. First we will argue that this setting is consistent with preserving all the supersymmetries from a four dimensional point of view. If we restrict ourselves to bosonic configurations $\psi_\mu = \chi = \lambda = 0$, the only non-vanishing supersymmetry transformations are the ones of the fermions given in (2.3-2.5). Furthermore, we consider the field configuration

$$\phi = \frac{\eta}{\sqrt{q}} e^{in\theta}, \quad A_\mu = n/(qe) \delta_{\mu\theta}, \quad (\text{A.5})$$

which ensures that the supersymmetry transformations of the chiralino χ and the gaugino λ are also zero. The only remaining supersymmetry transformation is the one of the gravitino (2.3), which can also be made zero provided the supersymmetry parameter ϵ satisfies

$$\mathcal{D}_\mu \epsilon_L = (\partial_\theta + \frac{pn}{q}) \epsilon_L = 0, \quad \implies \quad \epsilon_L(x^\mu) = e^{-i\frac{np}{q}\theta} \epsilon_L^0. \quad (\text{A.6})$$

From equation (2.3) it also follows that the supersymmetry parameter must satisfy the same boundary conditions as the gravitino, and thus it must be anti-periodic. Therefore the gravitino equation (A.6) admits solutions which are consistent with the boundary conditions provided the parameters satisfy the relation

$$\frac{|n|p}{q} = \frac{1}{2}. \quad (\text{A.7})$$

As $|p|, |n| \geq 1$, it is clear from this relation that supersymmetry can only be fully preserved when the charge of the chiral field satisfies $q \geq 2$, which justifies the inclusion of this parameter in (A.2). It is worth mentioning that the relation (A.7) is precisely the condition which ensures that the locally supersymmetric Abelian-Higgs model admits a critical cosmic string solutions of winding $n \in \mathbb{Z}$. Such solutions have a deficit angle of $\Delta = 2\pi$, and thus their background geometry asymptotes to $\mathbb{M}_3 \times S_1$ far away from the centre of the string, where supersymmetry is also fully preserved [9, 10]. The deficit angle of these strings solutions is given by (see [39])

$$\Delta = 2\pi |n| \frac{\eta^2 \kappa^2}{q} = 2\pi, \quad (\text{A.8})$$

which is equivalent to the condition (A.7) when the quantization of the FI term is taken into account. In the rest of the discussion we will assume for simplicity, and without loss of generality, that $n > 0$.

In [18, 19, 29] it was shown that non-periodic boundary conditions for the fermions, or a non-vanishing a vacuum expectation value of the gauge field, could induce masses of the order of the KK scale for the fermions, leading to the breaking of all the supersymmetries after the

dimensional reduction. However, when both situations occur simultaneously both effects may cancel each other, leading to a supersymmetric dimensionally reduced theory. We will now argue that this is precisely the situation we have at hand. Following [29], we will discuss the masses of the Kaluza-Klein modes, and in particular we show that the KK spectrum of all fermions contain light modes (with no contribution of the order of the KK scale) when the parameters satisfy the relation (A.7). This is a necessary condition for supersymmetry to remain unbroken in the reduced theory, as otherwise we would not have the right spectrum of particles to form the supermultiplets. The four dimensional fields can be expanded in series of Kaluza-Klein modes as follows:

$$\begin{aligned}
\phi &= \sum_{-\infty}^{\infty} \phi_m e^{im\theta}, \\
\chi_L &= e^{\frac{i}{2}\theta} \sum_{-\infty}^{\infty} \chi_{m|L} e^{im\theta}, \\
A_\mu &= \sum_{-\infty}^{\infty} A_{m|\mu} e^{im\theta}, \\
\lambda_L &= e^{-\frac{i}{2}\theta} \sum_{-\infty}^{\infty} \lambda_{m|L} e^{im\theta}, \\
\psi_{\mu L} &= e^{-\frac{i}{2}\theta} \sum_{-\infty}^{\infty} \psi_{m|\mu L} e^{im\theta},
\end{aligned} \tag{A.9}$$

where m is an integer labeling the KK modes, and the fields ϕ_m , $\chi_{m|L}$, etc... depend only on the non-compact coordinates $x^a \equiv (t, z, r)$. For A_μ to be real we also need $A_{m|\mu} = (A_{-m|\mu})^*$. Note that this ansatz for the KK expansion corresponds to a *generalised dimensional reduction* [18, 19], which ensures that the fermions satisfy anti-periodic boundary conditions, e.g. $\chi_L(x^a, \theta + 2\pi) = -\chi_L(x^a, \theta)$. The KK contribution to the masses is determined by the kinetic terms of the fields, and more specifically, by the form of the covariant derivatives along the compact θ coordinate of the S^1 . Taking into account the quantization of the FI-term the U(1) gauge transformations read

$$\begin{aligned}
\delta_g \phi &= i q e \phi \alpha, \\
\delta_g \chi_L &= i (q + p) e \chi_L \alpha, \\
\delta_g \psi_{\mu L} &= -i p e \psi_{\mu L} \alpha, \\
\delta_g \lambda_L &= -i p e \lambda_L \alpha,
\end{aligned} \tag{A.10}$$

where α is the U(1) gauge parameter, and thus the covariant derivatives along the θ direction have the form

$$\begin{aligned}
D_\theta \phi &= (\partial_\theta - i q e A_\theta) \phi, \\
D_\theta \chi_L &= (\partial_\theta - i e (q + p) A_\theta) \chi_L, \\
D_\theta \psi_{\mu L} &= (\partial_\theta + i e p A_\theta) \psi_{\mu L}, \\
D_\theta \lambda_L &= (\partial_\theta + i e p A_\theta) \lambda_L.
\end{aligned} \tag{A.11}$$

Note that, after taking into account the quantization of the FI-term, all the U(1) charges are integer multiples of the gauge coupling constant e . Using the expectation value of the gauge boson (A.5), we find that the KK modes have a contribution to the mass of the form¹⁶

$$\begin{aligned} M(\phi_m) &\sim |m - n| M_{\text{KK}}, \\ M(\chi_{m|L}) &\sim |m + \tfrac{1}{2} - n - \tfrac{np}{q}| M_{\text{KK}}, \\ M(\psi_{m|\mu L}) &\sim |m - \tfrac{1}{2} + \tfrac{np}{q}| M_{\text{KK}}, \\ M(\lambda_{m|L}) &\sim |m - \tfrac{1}{2} + \tfrac{np}{q}| M_{\text{KK}}, \end{aligned} \tag{A.12}$$

where the Kaluza-Klein mass scale is set by the radius of the compactification, $M_{\text{KK}} = R^{-1}$. Note that the contributions to the masses arising from the anti-periodicity of the fermions, the $1/2$ terms, are cancelled by the contribution associated to the background gauge field, np/q , provided the relation (A.7) is satisfied. It is now straightforward to check that the $m = n$ KK modes in the chiral multiplet, ϕ_n and $\chi_{n|L}$, do not receive contributions to the mass of the order of the KK scale M_{KK} . Similarly, in the gauge and graviton multiplets the $m = 0$ modes, $A_{0|\mu}$, $\lambda_{0|L}$ and $\psi_{0|\mu L}$, the Kaluza-Klein contributions to the mass are zero. This completes the consistency check showing that the KK spectrum contains the necessary light modes to form the supermultiplets of the reduced theory, and thus our results are fully compatible with the ones presented in [18, 19, 29].

B Numerical solutions

We obtained numerical solutions to the equations of motion by shooting from the bubble core outward, modifying the initial values until the desired asymptotic field values are achieved. In practice, this must be done independently over many adjacent intervals, where intermediate shooting parameters are introduced whose values are determined by continuity and smoothness at each junction. The full set of shooting parameters is solved for using Newton's method. This is called the multiple-shooting method. Regardless of how shooting is performed, a suitable initial (near core) and final (asymptotic) boundary condition must first be obtained.

B.1 Near core

Because our equations of motion are singular at the bubble (where $C = 0$), we will first Taylor expand all fields about $r = 0$, defined as where $C(r)$ vanishes, into generic form. A bubble solution without a conical singularity requires fixing the two coefficients

$$C(0) = 0, \quad C'(0) = 1, \tag{B.1}$$

so $C(r) = r + C''(0)r^2/2 + \dots$. The strongest singularities this introduces into the equations of motion (5.4) and (5.5) are

$$0 = \frac{\gamma (a'(0)^2 + 4a(0)^2 f(0)^2)}{2r^2} + \mathcal{O}(1/r), \tag{B.2}$$

$$0 = \frac{a(0)^2 f(0)}{r^2} + \mathcal{O}(1/r), \tag{B.3}$$

$$0 = \frac{-\gamma a'(0)^2}{2r^2} + \mathcal{O}(1/r), \tag{B.4}$$

¹⁶The fields in the chiral and gauge multiplets have additional contributions due to other interactions in the Lagrangian, as the scalar potential in the case of the chiral field ϕ .

assuming (as we do throughout this paper) a single winding number for the vortex namely, $n = 1$. Recall also that we work in a gauge where $B(0) = 1$. The relevant solution to these equations is $a'(0) = 0$, $f(0) = 0$. At next order, we obtain

$$0 = \frac{(a(0)^2 - 1) f'(0)}{r} + \mathcal{O}(r^0), \quad (\text{B.5})$$

$$0 = \frac{B'(0)}{r} + \mathcal{O}(r^0) \quad (\text{B.6})$$

which tells us that (since the sign of a is arbitrary)

$$a(0) = 1, \quad B'(0) = 0. \quad (\text{B.7})$$

At next order, we obtain the system of equations

$$B''(0) = \frac{\gamma(a''(0)^2 - \beta)}{4} + \frac{H_0^2}{2} + \mathcal{O}(r) \quad (\text{B.8})$$

$$C''(0) = 0 + \mathcal{O}(r) \quad (\text{B.9})$$

$$f''(0) = 0 + \mathcal{O}(r) \quad (\text{B.10})$$

Continuing order by order, we are left with three undetermined coefficients, H_0 , $f'(0)$, and $a''(0)$. All three of these should be thought of as shooting parameters, chosen to achieve the three boundary conditions for $r \rightarrow \infty$,

$$a \rightarrow 0, \quad f \rightarrow 1, \quad C \rightarrow R. \quad (\text{B.11})$$

Numerically, we can only integrate out to some finite $r = r_{\max}$, so we need to match the numerical solution there onto a suitable asymptotic solution.

B.2 Asymptotic solution

We can find an approximate asymptotic solution for $r_{\max} \leq r < \infty$ by linearizing the matter equations of motion about their vacuum values, yielding

$$a''(r) = 2a(r) - \left(\frac{2B'(r)}{B(r)} - \frac{C'(r)}{C(r)} \right) a'(r) \quad (\text{B.12})$$

$$f''(r) = 2\beta [f(r) - 1] - \left(\frac{2B'(r)}{B(r)} + \frac{C'(r)}{C(r)} \right) f'(r). \quad (\text{B.13})$$

These can be solved by the WKB method, since at large r the geometrical coefficients $\left(\frac{2B'(r)}{B(r)} \pm \frac{C'(r)}{C(r)} \right)$ are small compared to the masses $m_a = \sqrt{2}$, $m_f = \sqrt{2\beta}$. By writing $a = \exp(\log a)$ and using the WKB approximation to drop second derivatives of $\log a$, we find the second-order equations are well-approximated by the first-order equations

$$a'(r) = - \left[\sqrt{2 + \left(\frac{B'(r)}{B(r)} - \frac{C'(r)}{2C(r)} \right)^2} + \left(\frac{B'(r)}{B(r)} - \frac{C'(r)}{2C(r)} \right) \right] a(r), \quad (\text{B.14})$$

$$f'(r) = - \left[\sqrt{2\beta + \left(\frac{B'(r)}{B(r)} + \frac{C'(r)}{2C(r)} \right)^2} + \left(\frac{B'(r)}{B(r)} + \frac{C'(r)}{2C(r)} \right) \right] [f(r) - 1], \quad (\text{B.15})$$

where the signs of the square roots are chosen by the boundary conditions at $r = \infty$. Of more immediate use, these equations provide an excellent matter boundary condition for finite $r = r_{\max}$, which allows us to use a shooting method to construct the numerical solutions. The third boundary condition comes from the (vacuum) constraint equation (4.6), which implies

$$C(r_{\max}) = R B'(r_{\max})/H_{\delta}. \quad (\text{B.16})$$

These three relations (B.14-B.16) are the practical versions of equation (B.11).

From equations (B.14-B.15) it is clear that the matter fields will approach their vacuum values exponentially quickly. Far from the vortex we can trust the the vacuum Einstein equations, which imply

$$\begin{aligned} r(B) &= r_{\delta} + H_{\delta}^{-1} \sqrt{B(B-1)} + H_{\delta}^{-1} \log \left(\sqrt{B} + \sqrt{B-1} \right), \\ C(r) &= R \sqrt{1 - B(r)^{-1}}, \end{aligned} \quad (\text{B.17})$$

where the parameter

$$H_{\delta} = B'(r_{\max})/\sqrt{1 - B(r_{\max})^{-1}}, \quad (\text{B.18})$$

is the Hubble parameter of the corresponding pure vacuum bubble of nothing, which would have a delta-function singularity at

$$r_{\delta} = r_{\max} - H_{\delta}^{-1} \sqrt{B(B-1)}|_{r_{\max}} - H_{\delta}^{-1} \log \left(\sqrt{B} + \sqrt{B-1} \right) \Big|_{r_{\max}}. \quad (\text{B.19})$$

This means that having integrated the solution numerically to a large enough r_{\max} , we can read off the parameters of the vacuum solution directly from the numerical values of the functions at $r = r_{\max}$. This method allows us to obtain r_{δ} and H_{δ} , and from there we can get the more physical parameter, the deficit angle Δ_W . These are related through R by

$$H_{\delta} = \frac{2\pi - \Delta_W}{\pi R}, \quad (\text{B.20})$$

where Δ_W is the deficit angle (relative to Witten's bubble of nothing), measured at $r = \infty$. This is the way we obtain the values of Δ_W that we present in our numerical solutions and that we compare with the analytic estimates based on the arguments of section 4.2.

References

- [1] S. R. Coleman, *The Fate of the False Vacuum. 1. Semiclassical Theory*, *Phys. Rev.* **D15** (1977) 2929–2936. [Erratum: *Phys. Rev.* **D16**, 1248(1977)].
- [2] C. G. Callan, Jr. and S. R. Coleman, *The Fate of the False Vacuum. 2. First Quantum Corrections*, *Phys. Rev.* **D16** (1977) 1762–1768.
- [3] S. R. Coleman and F. De Luccia, *Gravitational Effects on and of Vacuum Decay*, *Phys. Rev.* **D21** (1980) 3305.
- [4] J. J. Blanco-Pillado, R. Kallosh, and A. D. Linde, *Supersymmetry and stability of flux vacua*, *JHEP* **05** (2006) 053, [[hep-th/0511042](#)].
- [5] M. Cvetič, S. Griffies, and S.-J. Rey, *Nonperturbative stability of supergravity and superstring vacua*, *Nucl. Phys.* **B389** (1993) 3–24, [[hep-th/9206004](#)].

- [6] E. Witten, *Instability of the Kaluza-Klein Vacuum*, *Nucl.Phys.* **B195** (1982) 481.
- [7] M. Dine, P. J. Fox, and E. Gorbatov, *Catastrophic decays of compactified space-times*, *JHEP* **09** (2004) 037, [[hep-th/0405190](#)].
- [8] V. A. Rubakov and M. Yu. Kuznetsov, *Fermions and Kaluza-Klein vacuum decay: a toy model*, *Theor. Math. Phys.* **175** (2013) 489–498, [[arXiv:1205.5184](#)].
- [9] J. D. Edelstein, C. Nunez, and F. A. Schaposnik, *Supergravity and a Bogomolny bound in three-dimensions*, *Nucl. Phys.* **B458** (1996) 165–188, [[hep-th/9506147](#)].
- [10] G. Dvali, R. Kallosh, and A. Van Proeyen, *D term strings*, *JHEP* **0401** (2004) 035, [[hep-th/0312005](#)].
- [11] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 2012.
- [12] C. J. Isham, *Twisted Quantum Fields in a Curved Space-Time*, *Proc. Roy. Soc. Lond.* **A362** (1978) 383–404.
- [13] S. J. Avis and C. J. Isham, *Vacuum solutions for a twisted scalar field*, *ICTP-77-78/9a*, *ICTP-77-78/14*, (1978).
- [14] C. J. Isham, *Spinor Fields in Four-dimensional Space-time*, *Proc. Roy. Soc. Lond.* **A364** (1978) 591–599.
- [15] S. J. Avis and C. J. Isham, *Quantum field theory and fiber bundles in a general space-time*, in *In *Cargese 1978, Proceedings, Recent Developments In Gravitation**, 347–401, 1978.
- [16] B. S. DeWitt, C. F. Hart, and C. J. Isham, *Topology and quantum field theory*, *Physica* **A96** (1979) 197–211.
- [17] M. Nakahara, *Geometry, topology and physics*, Boca Raton, USA: Taylor and Francis (2003) 573 p (2003).
- [18] J. Scherk and J. H. Schwarz, *Spontaneous Breaking of Supersymmetry Through Dimensional Reduction*, *Phys. Lett.* **B82** (1979) 60.
- [19] J. Scherk and J. H. Schwarz, *How to Get Masses from Extra Dimensions*, *Nucl. Phys.* **B153** (1979) 61–88.
- [20] T. Ortin, *Gravity and Strings*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2015.
- [21] L. H. Ford, *Vacuum Polarization in a Nonsimply Connected Space-time*, *Phys. Rev.* **D21** (1980) 933.
- [22] L. H. Ford, *Twisted Scalar and Spinor Strings in Minkowski Space-time*, *Phys. Rev.* **D21** (1980) 949–957.
- [23] A. Ashtekar and V. Petkov, eds., *Springer Handbook of Spacetime*. Springer, Berlin, 2014.
- [24] M. Henneaux, *Energy momentum, angular momentum, and supercharge in 2+1 Supergravity*, *Phys. Rev.* **D29** (1984) 2766–2768.
- [25] P. S. Howe, J. M. Izquierdo, G. Papadopoulos, and P. K. Townsend, *New supergravities with central charges and Killing spinors in (2+1)-dimensions*, *Nucl. Phys.* **B467** (1996) 183–214, [[hep-th/9505032](#)].
- [26] E. Witten, *Is supersymmetry really broken?*, *Int. J. Mod. Phys.* **A10** (1995) 1247–1248, [[hep-th/9409111](#)].
- [27] S. Forste and A. Kehagias, *Zero branes in (2+1)-dimensions*, in *Gauge theories, applied supersymmetry and quantum gravity. Proceedings, 2nd Conference, London, UK, July 5-10, 1996*, pp. 263–270, 1996. [hep-th/9610060](#).

- [28] K. Becker, M. Becker, and A. Strominger, *Three-dimensional supergravity and the cosmological constant*, *Phys. Rev.* **D51** (1995) 6603–6607, [[hep-th/9502107](#)].
- [29] Y. Hosotani, *Dynamical Mass Generation by Compact Extra Dimensions*, *Phys. Lett.* **B126** (1983) 309.
- [30] J. J. Blanco-Pillado and B. Shlaer, *Bubbles of Nothing in Flux Compactifications*, *Phys. Rev.* **D82** (2010) 086015, [[arXiv:1002.4408](#)].
- [31] J. J. Blanco-Pillado, H. S. Ramadhan, and B. Shlaer, *Decay of flux vacua to nothing*, *JCAP* **1010** (2010) 029, [[arXiv:1009.0753](#)].
- [32] J. J. Blanco-Pillado, H. S. Ramadhan, and B. Shlaer, *Bubbles from Nothing*, *JCAP* **1201** (2012) 045, [[arXiv:1104.5229](#)].
- [33] I.-S. Yang, *Stretched extra dimensions and bubbles of nothing in a toy model landscape*, *Phys. Rev.* **D81** (2010) 125020, [[arXiv:0910.1397](#)].
- [34] A. R. Brown and A. Dahlen, *Bubbles of Nothing and the Fastest Decay in the Landscape*, *Phys. Rev.* **D84** (2011) 043518, [[arXiv:1010.5240](#)].
- [35] S. de Alwis, R. Gupta, E. Hatefi, and F. Quevedo, *Stability, Tunneling and Flux Changing de Sitter Transitions in the Large Volume String Scenario*, *JHEP* **11** (2013) 179, [[arXiv:1308.1222](#)].
- [36] A. Vilenkin and E. Shellard, *Cosmic strings and other topological defects*, Cambridge University Press, Cambridge (1994).
- [37] F. Englert, L. Houart, and P. Windey, *The Black hole entropy can be smaller than $A/4$* , *Phys. Lett.* **B372** (1996) 29–33, [[hep-th/9503202](#)].
- [38] F. Englert, L. Houart, and P. Windey, *Black hole entropy and string instantons*, *Nucl. Phys.* **B458** (1996) 231–248, [[hep-th/9507061](#)].
- [39] J. J. Blanco-Pillado, B. Reina, K. Sousa, and J. Urrestilla, *Supermassive Cosmic String Compactifications*, *JCAP* **1406** (2014) 001, [[arXiv:1312.5441](#)].
- [40] J. Distler and E. Sharpe, *Quantization of Fayet-Iliopoulos Parameters in Supergravity*, *Phys. Rev.* **D83** (2011) 085010, [[arXiv:1008.0419](#)].
- [41] N. Seiberg, *Modifying the Sum Over Topological Sectors and Constraints on Supergravity*, *JHEP* **1007** (2010) 070, [[arXiv:1005.0002](#)].